Solving a Relaxed Min-Cost Redundancy Allocation Model with a Lagrange Multiplier and Newton's Method

Abstract

Redundancy allocation is a valuable technique that system engineers can use to design high level of reliability into complex systems. Broadly however, redundancy allocation problems are NP-hard. The main goal of this paper is to solve a relaxed minimum-cost problem by proving that Newton's method finds the optimal value of the Lagrange multiplier.

Keywords: Redundancy, redundancy allocation, redundancy allocation models, reliability, NP-hard, discrete problem, Lagrange multipliers, Newton's method, series-parallel systems, optimization models

1 Introduction

The desire to improve the reliability of products or complex systems in a competitive market has been of paramount interest to industries (Leon and Cascaval [1]). As Rice et al [2] and Kuo and Wan [3] pointed out, redundancy allocation can be used to improve the reliability of a structure with inadequate reliability. In fact, since the birth of the industrial revolution, manufacturers have been finding ways to build trust in their products. But it was not until after the Second World War that the scholarly foundations of reliability were developed (Japan Standard Association [4] and Tillman et al [5]).

To meet this desired need, industrial system designers have turned to redundancy allocation techniques (Rice et al [2], Tillman et al [6], and Devi et al [7]). Redundancy allocation is a useful and practical technique that engineering designers use when designing engineering systems that need high levels of system reliability while satisfying limitations on cost, weight, volume, etc.(Rice et al [2]). It is done during the design phase where system designers install additional identical redundant components arranged in parallel. Thus, redundancy allocation models may be viewed as system designs in which system engineers built component redundancies into the system. Both Elsayed [8] and Chern [9] pointed out that, in general, redundancy allocation models are NP-hard.

One of the initial methods for solving optimal redundancy allocation problems was based on the classical Lagrange multiplier method (Ushakov [10]). Li and Zio [11], Devi et al [7] as well as Tillman et al [5], respectively, gave in-depth and comprehensive literature reviews of various redundancy allocation models. Both Li et al [11] and Leon et al [12] considered several optimization methods for redundancy allocation in large systems and did an excellent study of series-parallel reliability models. Recently, the use of redundancy allocation models has entered into the space of artificial intelligence Devi et al [7].

This paper deals with non-linear optimization problems of the form

$$\min \sum_{i=1}^{n} c_i x_i \tag{1}$$

$$\text{subject to } \prod_{i=1}^{n}(1-\rho_i^{x_i}) \geq R \text{ and } x_i > 0,\ i = 1, \ldots, n.$$

The size of the model, n, is an integer greater than 1 and the values of the decision variables, $x_1, x_x, \ldots, x_n$, are restricted to positive real numbers. The objective coefficients are positive real numbers $c_1, c_2, \ldots, c_n$. Since the objective is linear, it can be scaled so that $\min_{1\leq i\leq n} c_i = 1$. The constraint parameters are real numbers, R, the predetermined system reliability, and $\rho_i,\ i = 1, 2, 3, \cdots, n$, the unreliability component at the i^{th}stage, for which $0 < R < 1$ and $0 < \rho_i < 1$ for $i = 1, 2, \ldots, n$.

Model (1) can be solved by the method of Lagrange multipliers. Nmah ([13] and [14]) showed that there is a unique optimal solution $(x_1^*, x_2^*, \ldots, x_n^*)$ of the form

$$x_i^* = \ln(c_i/(c_i + \lambda^* R \ln(\rho_i^{-1})))/\ln(\rho_i), \tag{2}$$

where λ^* is the unique positive root of the equation

$$\prod_{i=1}^{n} \lambda R ln(\rho_i^{-1})/(c_i + \lambda R \ln(\rho_i^{-1})) = R. \tag{3}$$

The classic redundancy allocation model is a discrete optimization model that differs from (1) only in that the values of the decision variables, x_i, are restricted to positive integers. In the context of redundancy allocation, the optimal values of the decision variables for the discrete optimization model show the levels of redundancy for each subsystm in a series system that will achieve a required level of reliability, while minimizing a competing characteristic, such as cost or weight.

The purpose of this work is to show that Equation (3) can be solved by Newton's method from an initial value determined by the model's parameters. The main result of Section 2 gives upper and lower bounds for the root λ^*. The results of Section 3 show how to approximate λ^* by Newton's method, and Section 4 presents examples and conclusions.

2 Upper and Lower Bounds on the Optimal Lagrange Multiplier

Constants $a_1, a_2, \ldots, a_n$ can be computed from the parameters c_i, ρ_i, and R of Model (1) with the equations

$$a_i = c_i/R\ln(\rho_i^{-1}). \tag{4}$$

These strictly positive constants determine a rational function f defined on $[0, \infty)$ as

$$f(\lambda) = \prod_{i=1}^{n} \lambda/(a_i + \lambda). \tag{5}$$

When Equation (3) is rewritten in terms of $a_1, a_2, \ldots, a_n$, the result is

$$f(\lambda^*) = R. \tag{6}$$

Let a_{min} and a_{max} denote the smallest and largest of the constants in Equation (4) and let the constants λ_{min} and λ_{max} be given by the equations

$$\lambda_{min} = a_{min}\frac{R^{1/n}}{1 - R^{1/n}} \quad \text{and} \quad \lambda_{max} = a_{max}\frac{R^{1/n}}{1 - R^{1/n}}.$$

Proposition 1 *The optimal Lagrange multiplier λ^* satisfies the inequalities $\lambda_{min} \leq \lambda^* \leq \lambda_{max}$.*

Proof. The function f is strictly increasing on $[0, \infty)$ and its range is the interval $[0, 1)$. Since $0 < R < 1$, Equation (6) has a unique, strictly positive solution. Clearly, $0 < \lambda_{min} \leq \lambda_{max}$ and

$$\left(\frac{\lambda_{min}}{a_{min} + \lambda_{min}}\right)^n = R = \left(\frac{\lambda_{max}}{a_{max} + \lambda_{max}}\right)^n.$$

Since $f(\lambda) \leq (\lambda/(a_{min} + \lambda))^n$, it follows that $f(\lambda_{min}) \leq R$ and $\lambda_{min} \leq \lambda^*$. Likewise, $f(\lambda) \geq (\lambda/(a_{max} + \lambda))^n$, so $\lambda_{max} \geq \lambda^*$. ■

Corollary 1 *Either $a_{min} = a_{max}$, in which case $\lambda^* = \lambda_{min}$ and $f(\lambda_{min}) = R$, or $a_{min} < a_{max}$, in which case $\lambda^* > \lambda_{min}$, and $f(\lambda_{min}) < R$.*

Proof. If $a_{min} = a_{max}$, then the proposition shows that $\lambda^* = \lambda_{min}$ and so $f(\lambda_{min}) = R$. If $a_{min} < a_{max}$, then $f(\lambda) < (\lambda/(a_{min} + \lambda))^n$ on $(0, \infty)$. In particular, $f(\lambda_{min}) < R$, so $\lambda^* > \lambda_{min}$. ■

Corollary 2 *For ρ and R in $(0, 1)$, set $c_i = 1$ and $\rho_i = \rho$ for $1 \leq i \leq n$. Then the solution of Model (1) is*

$$\lambda^* = R^{1/n} / \ln(\rho^{-1}) R(1 - R^{1/n}),$$

and

$$x_i^* = \ln(1 - R^{1/n}) / \ln(\rho) \quad \textit{for} \quad 1 \leq i \leq n.$$

Proof. For these parameters, $a_{min} = a_{max} = 1/R \ln(\rho^{-1})$. Then, the optimal value λ^* is given by the previous proposition and the optimal values of the coordinates x_i^* are given by Equation (2). ■

3 Using Newton's Method

The purpose of this section is to establish that Newton's method, applied to a transformation of the function f of Equation (5) and initialized with λ_{min}, generates an increasing sequence that converges to the optimal Lagrange multiplier.

Definition 1 *For the constants $a_1, a_2, \ldots, a_n$, of Equation (4), the function h is defined on $(0, \infty)$ as*

$$h(\lambda) = \ln(f(\lambda)) - \ln(R) = \sum_{i=1}^{n} \ln(\lambda/(a_i + \lambda)) - \ln(R).$$

Lemma 1 *Let h be as defined on $(0, \infty)$ in Definition 1. Then* $h^{(k)}(\lambda) = (-1)^{k-1}(k-1)! \sum_{i=1}^{n} \left(\frac{1}{\lambda^k} - \frac{1}{(a_i + \lambda)^k} \right)$, $k = 1, 2, 3, \cdots$.

Proof. The proof follows from the Principle of Mathematical Induction. To begin, let $k = 1$; then:

$$\begin{aligned} h'(\lambda) &= \frac{d}{d\lambda} \left(\sum_{i=1}^{n} \ln\left(\frac{\lambda}{(a_i + \lambda)} \right) - \ln(R) \right) \\ &= \sum_{i=1}^{n} \frac{d}{d\lambda} (\ln(\lambda) - \ln(a_i + \lambda)) \\ &= \sum_{i=1}^{n} \left(\frac{1}{\lambda} - \frac{1}{a_i + \lambda} \right) \end{aligned}$$

Now assume that it is true for k. That is,
$h^{(k)}(\lambda) = (-1)^{k-1}(k-1)! \sum_{i=1}^{n} \left(\frac{1}{\lambda^k} - \frac{1}{(a_i+\lambda)^k} \right).$
Then for $k+1$, we have $h^{(k+1)}(\lambda) = \frac{d}{d\lambda} h^{(k)}(\lambda),$
which by the induction hypothesis is equivalent to:

$$\begin{aligned}
h^{(k+1)}(\lambda) &= \frac{d}{d\lambda}\left((-1)^{k-1}(k-1)! \sum_{i=1}^{n} \left(\frac{1}{\lambda^k} - \frac{1}{(a_i+\lambda)^k} \right) \right) \\
&= (-1)^{k-1}(k-1)! \frac{d}{d\lambda}\left(\sum_{i=1}^{n} \left(\frac{1}{\lambda^k} - \frac{1}{(a_i+\lambda)^k} \right) \right) \\
&= (-1)^{k-1}(k-1)!(-k) \sum_{i=1}^{n} \left(\frac{1}{\lambda^{k+1}} - \frac{1}{(a_i+\lambda)^{k+1}} \right) \\
&= (-1)^k k! \sum_{i=1}^{n} \left(\frac{1}{\lambda^{k+1}} - \frac{1}{(a_i+\lambda)^{k+1}} \right). \blacksquare
\end{aligned}$$

Note: While all we need is $h \in C^2((0,\infty))$, it is clear from Lemma 1 that $h \in C^\infty((0,\infty))$.

Lemma 2 *The function h is strictly increasing on $(0,\infty)$ and strictly concave there.*

Proof. On $(0,\infty)$, the derivative, h', is strictly positive and the second derivative, h", is strictly negative. In fact, Lemma 1 with $k = 1, 2$ yields $h'(\lambda) = \sum_{i=1}^{n} \left(\frac{1}{\lambda} - \frac{1}{a_i+\lambda} \right)$ and $h''(\lambda) = -\sum_{i=1}^{n} \left(\frac{1}{\lambda^2} - \frac{1}{(a_i+\lambda)^2} \right)$.$\blacksquare$

Corollary 3 *The optimal Lagrange multiplier λ^* is the unique zero of the function h.*

Proof. That $h(\lambda^*) = 0$ follows from the definition of the function h. This root is unique because h is strictly increasing and its range is $(-\infty, 0)$. ■

Definition 2 *When $a_{min} < a_{max}$, the sequence λ_k is generated by the equations* $\lambda_0 = \lambda_{min}$ *and* $\lambda_{k+1} = \lambda_k - h(\lambda_k)/h'(\lambda_k)$.

When $a_{min} = a_{max}$, Corollary 1 shows that no iterative method is needed to find the root λ^*. When $a_{min} < a_{max}$, the proof of Theorem 1 will show that each term of the sequence $\{\lambda_k : k > 0\}$ is positive and thus in the domain of the functions h and h'.

The proofs of the next two theorems are adaptations of an argument in Allen & Isaacson [15], where it is attributed to Henrici [16].

Theorem 1 *If $a_{min} < a_{max}$, then, for $k > 0$, $0 < \lambda_{min} < \lambda_k < \lambda_{k+1} < \lambda^*$.*

Proof. Suppose $a_{min} < a_{max}$. Then Corollary 1 shows that $\lambda_0 < \lambda^*$, and so $h(\lambda_0) < 0$. Furthermore, since $h'(\lambda_0) > 0$ and $\lambda_1 = \lambda_0 - h(\lambda_0)/h'(\lambda_0)$, it follows that $\lambda_1 > \lambda_0$.

Next, we need to show that $\lambda_1 < \lambda^*$. Since $\lambda_0 < \lambda^*$, the Mean Value Theorem implies that $-h(\lambda_0) = h(\lambda^*) - h(\lambda_0) = h'(c)(\lambda^* - \lambda_0)$ for some $c \in (\lambda_0, \lambda^*)$. From Lemma 2, h' is strictly decreasing, so $h'(c) < h'(\lambda_0)$. Thus, $-h(\lambda_0) < h'(\lambda_0)(\lambda^* - \lambda_0)$, or $\lambda_1 - \lambda_0 < \lambda^* - \lambda_0$.

Next, assume that $\lambda_0 < \lambda_k < \lambda^*$. As above, it follows that $\lambda_{k+1} > \lambda_k$. From the Mean Value Theorem and the fact that h' is decreasing, we have $-h(\lambda_k) < h'(\lambda_k)(\lambda^* - \lambda_k)$, or $\lambda_{k+1} - \lambda_0 < \lambda^* - \lambda_0$. ■

As noted by Allen & Isaacson [15] and Henrici [16], the next theorem relies on a result from the theory of real variables which states that a bounded, nondecreasing sequence of real numbers converges to its least upper bound (Lebl [17]).

Theorem 2 *If* $a_{min} < a_{max}$, *then* $\lim_{k\to\infty} \lambda_k = \lambda^*$.

Proof. Suppose $a_{min} < a_{max}$. Then by Theorem 1, $\{\lambda_k\}$ is a bounded, nondecreasing sequence of real numbers. Thus, it has a limit, say $\bar{\lambda}$. From Theorem 1, it also follows that $0 < \lambda_{min} < \bar{\lambda} \leq \lambda^*$ and so $\bar{\lambda}$ is in the domain of both h and h'. Then since h and h' are continuous and h' is strictly positive on $(0, \infty)$, taking limits, as in Henrici [14], on both sides gives $\bar{\lambda} = \bar{\lambda} - h(\bar{\lambda})/h'(\bar{\lambda})$ or $h(\bar{\lambda}) = 0$. By Corollary 3, λ^* is the only zero of h, and so, $\bar{\lambda} = \lambda^*$. ■

4 Examples

The main result of this paper is to solve a relaxed minimum-cost redundancy allocation problem by showing that Newton's method finds the value of the optimal Lagrange multiplier.

Elsayed [8] posed a problem with $n = 3$, $c_1 = c_2 = c_3 = 1$, minimum required system reliability, $R = 0.82$, and component unreliabilities $\rho_1 = 0.30$, $\rho_2 = 0.25$, and $\rho_3 = 0.15$. Table 1 displays the solution to the Lagrange multiplier problem, using Newton's method.

Table 1 Computing the Optimal Lagrange multiplier

k	λ_k	$\lvert\lambda_{k+1} - \lambda_k\rvert$		k	λ_k	$\lvert\lambda_{k+1} - \lambda_k\rvert$
0	9.399742			3	12.34503257	0.03609972
1	11.665351	2.265609		4	12.34505203	1.946×10^{-5}
2	12.30893285	0.64358185				

Now, using $\lambda^* = 12.34505203$ and Equation (2), we calculate the unique optimal solution x^* to the continuous relaxation model to be (2.14231433, 1.955048673, 1.584455517).

Note: In general, rounding coordinates of the optimal solution of the relaxed model to the nearest integer does not give a feasible solution. Nmah([13] and [18]) showed that the lower and upper bounds on the optimal value of the discrete model are $\lceil c^T x^* \rceil$ and $c^T x^* + \sum_{i=1}^{3} c_i$, respectively, and that any integral optimal solution of the discrete model lies in an intersection ,

$\{x : c^T x = \beta, x \geq 1\}$, of a hyperplane and the positive cone, where β is an integer between $c^T x^*$, and $c^T x^{(*I)}$, and where $x_i^{*I} = \lceil x_i^* \rceil$. In fact, Nmah [13] argued that in the absence of such solutions, $x^{(*I)}$ becomes the integral optimal solution to the discrete model. Specifically, the lower and upper bounds on the optimal value of the discrete model are 6 and 8.6818152, respectively. Since in this case there is no integer between $c^T x^*$, and $c^T x^{(*I)}$, we conclude that (3, 2 2) is the integral optimal solution to the discrete model.

Rice et al [2] posed a problem with $n = 3, R = 0.995$, $\rho_1 = 0.1$, $\rho_2 = 0.125$, $\rho_3 = 0.09$, $c_1 = 4$, $c_2 = 1$, and $c_3 = 3$. Table 2 displays the results of the Lagrange multiplier problem, using Newton;s method.

Table 2 Computing the Optimal Lagrange multiplier

k	λ_k	$\|\lambda_{k+1} - \lambda_k\|$		k	λ_k	$\|\lambda_{k+1} - \lambda_k\|$
0	289.0217848			4	693.7101289	9.1278919
1	457.8854805	168.8636957		5	693.8333318	0.1232029
2	613.6762507	155.74777026		6	693.8333537	2.19×10^{-5}
3	684.582237	70.9059863				

Now, using $\lambda^* = 693.8333537$ and Equation (2), we calculate the unique optimal solution x^* to the continuous relaxation model to be (2.600325403, 3.496138185, 2.624305105). Applying the method in Nmah ([13] and [18]), we see that the lower and upper bounds on the optimal

value of the discrete model are 22 and 29.77035511, respectively, and that any integral optimal solution of the discrete model lies in an intersection, $\{x : c^T x = \beta, x \geq 1\}$, of a hyperplane and the positive cone, where β is an integer between 21.770355511 and 25. Since there are three integers between 21.770355511 and 25, we have only three hyperplanes,
$\{x : c^T x = \beta, x \geq 1, \beta \in \{22, 23, 24\}\}$ to search. Searching these three hyperplanes and using the strategy in Nmah [19], we get $(3, 3, 3)$ as the integral optimal solution to the discrete model.

Note: We observe that Rice et al [2] achieved the same result by a different method.

Conclusion

The main result of this work is to solve a relaxed minimum-cost problem by showing that from an initial value determined by the model's parameters, Newton's method finds the optimal value, λ^*, of the Lagrange multiplier. In Section 2 we found upper and lower bounds for the roots, λ^*, of equation (3), and proved that the optimal Lagrange multiplier λ^*, satisfies $\lambda_{min} \leq \lambda^* \leq \lambda_{max}$. Nmah ([13] and [14]) proved that the vector x^* and the positive multiplier λ^* that satisfy equations (2) and (3) are unique, and we showed in Section 3 that the optimal Lagrange multiplier λ^* is the unique zero of the $C^2((0, \infty))-$function h defined in Section 3. Also in Section 3 we showed how to approximate λ^* by Newton's method. Finally, in this Section we used two examples to illustrate the method.

References

[1] Leon F, Cascaval P. Optimization methods for redundancy allocation in complex system. 23rd International Conference on System Theory, Control and Computing, 2019

[2] Rice W., Cassady C., Wise T. Simplifying the solution of redundancy allocation problems, 1999 Proceedings Annual Reliability and Maintainability Symposium, , pp190-194

[3] Kuo W., Wan, R. Recent advances in optimal reliability allocation, Coput Intelli in Reliability Engineering,Springer Berlin, 2007, pp1-36

[4] Japan Standard Association. Reliability guidebook, Asian Productivity Organization, 1972

[5] Tillman F., Hwang C., Kuo W. Optimization techniques for system reliability with redundancy: A review, *IEEE Transactions on Reliability, Vol R-26, No. 3, Aug 1977, 148-155*

[6] Tillman F, Hwang C., Kuo W. Optimization of system reliability, Marcel Dekker, INc. 1980

[7] Devi S. Garg H., Garg D. A review of redundancy allocation problems for two decades, bibliometrics and future directions, Artificial Intelligence Review, 2023, 56, 7457-7548

[8] Elsayed E.A., Reliability Engineering, 3rd edition, John Wiley & Sons, Inc., 2021

[16] Chern MS. On the computational complexity of reliability redundancy allocation in series systems, Operation Research Letters, 1992; 11:309-315

[10] Ushakov I. Optimal resource allocation: with practical statistical applications and theory, First Edition, John Wiley & Sons, Inc, 2013

[11] Li Y, Zio E, Kleyner A. System reliability assessment and optimization: methods and applications (e-book), John Wiley & Sons, Inc, 2022

[12] Leon F., Cascaval P., Badica C. Optimization methods for redundancy allocation in large sysyems, Vietnam Journal of Computer Science, Vo 7, No. 3 (2020) 281-299

[13] Nmah B. Optimizing system reliability with integer programming, New Mexico State University: UMI Dissertation Services, Bell & Howell Company, 1997

[14] Nmah B, Optimal allocation of redundancy in a mixed system, International Journal of Pure and Applied Mathematics, Vol 73, No. 3, 2011, 255-265

[15] Allen III, B, Isaacson L. Numerical analysis for applied science, John Wiley & Sons, Inc.,2019

[16] Henrici P. Elements of numerical analysis, John Wiley & Sons, Inc., 1964

[17] Lebl J., Introduction to real analysis, Volume I, CreateSpace Independent Publishing Platform, 2018. ISBN 9781718862401

[18] Nmah B. A computational complexity analysis of an algorithm for reliability optimization. British Journal of Mathematics and & Computer Science. 2015; 7(1):18-25

[19] Nmah B. Shorter searches for least-cost allocation of redundancy.

American Journal of Scientific and Industrial Research. 2015; 6(6):116-122.