Dynamic Behaviour of Simply Supported Non-Uniform Rayleigh Beam under Variable-Magnitude Accelerating Masses and Resting on Non-Uniform Bi-parametric Foundation

Abstract

The dynamic behaviours of simply supported non-uniform Rayleigh beam under variable-magnitude accelerating masses and resting on non-uniform bi-parametric foundation is investigated in this paper. A partial differential equation of fourth order governs the situation. The governing equation is converted into a series of coupled second order ordinary differential equations with variable coefficients using the Galerkin approach, which is based on the series representation of the Heaviside function.Two instances are examined; (i) the moving force problem when the inertia term is neglected and (ii) the moving mass case when the inertia term is considered. Variation of parameters are employed to get the transverse displacement response in order to solve the moving force problem,the moving mass problem cannot be solved using the widely used Struble's asymptotic method due to the variability of the load magnitude. Therefore, a numerical technique, specifically the Runge-Kutta of fourth order is used to obtain an approximate solution. The numerical solution of the moving force problem is compared with the analytical solution in order to verify the accuracy of the Runge-Kutta scheme, and it compares favorably. From the analytical and numerical result, it is observed that the amplitude of the deflection profile of simply supported non-uniform Rayleigh beam decreases with an increase in the value of some vital structural parameters such as rotatory inertia correction factor, axial force, shear modulus and foundation modulus.

Keywords: *Galerkin's method, non-uniform beam, bi-parametric foundation , accelerating Masses and variation of parameter.*

1 Introduction

In structural engineering and mechanical systems, the study of beam dynamics is essential, especially when dealing with complex loading and foundation conditions. Because it takes into account the effects of both rotating inertia and shear deformation, the Rayleigh beam theory has become one of the most popular beam models and a flexible framework for understanding beam behavior. The investigation of dynamic responses exhibited by structural members resting upon elastic foundations when subjected to moving loads holds significant interest and importance. Certain outcomes from this study can be applied to enhance our understanding of the dynamic characteristics of roadways, aircraft, bridges, aircraft and machinery where vibration and deflection play a critical roles. Noteworthy contributions to this field can be found in the works of [1], [2], [3], [4], [5] and [6]. Remarkable advancements in the study of structures under moving loads have been made by researchers such as [7] and [8].

However, all the researchers mentioned considered only one parameter foundation model called winkler foundation often used in pavement modelling. Since the characteristic features of the popular Winkler foundation model is the discontinous behaviour of the surface displacement beyond the load layer, a more realistic elastic foundation model known as the bi-parametric foundation model is considered in this paper. This foundation model provides a comprehensive understanding of soil structures interaction, making it an essential tool for engineers and researchers working on a complex projects, such as offshores structures, high-rise buildings and geotechnical engineering . The dynamic behaviour of beams on foundations, exposed to moving loads of varying magnitudes, presents a plethora of complexities that have been extensively explored by researchers in engineering, applied mathematics, mathematical physics, particularly in the domains of railway engineering and construction engineering. The rapid expansion of high-speed railway networks has further propelled research efforts aimed at accurately predicting the vibration tendencies of railway tracks. Pioneering work in the field of problems involving variable speeds was undertaken by [9] ,who addressed the transverse oscillations of beams subjected to moving variable loads. Afterwards, [10] examined the dynamic response of finite beams with continuously applied visco-elastic foundations when loads are moving in different directions. Notably, idealized models of concentrated loads that act at particular points along a single line in space were adopted in these investigations, which frequently only took into account the force impacts of moving loads [3]. However, it is recognized that loads are distributed over small segments or the entire length of the structural member as they traverse it up to the present moment, scanty attention has been directed towards scenarios involving beams on non-uniform bi-parametric foundations exposed to moving loads with variable magnitudes. This can be attributed to the intricacies associated with the model's complexity and the challenges posed in estimating parameter values, particularly when dealing with a non-uniform bi-parametric foundation. Therefore, this paper is devoted to exploring the dynamic behavior of non-uniform simply supported Rayleigh beams when subjected to variable-magnitude accelerating masses, while being supported by non-uniform bi-parametric foundations. Through this investigation, we aim to shed light on the dynamics underlying such scenarios.

2 Mathematical Model

Consideration is given to the flexural vibrations of a non-uniform simply supported Rayleigh beam sitting on a bi-parametric foundation and subjected to a variable-magnitute accelerating load. The fourth order partial differential equation [11] is the corresponding governing equation.

$$\frac{\partial^2}{\partial x^2}\left[EI(x)\frac{\partial^2 Y(x,t)}{\partial x^2}\right] - N\frac{\partial^2 Y(x,t)}{\partial x^2} + \mu(x)\frac{\partial^2 Y(x,t)}{\partial t^2} - \mu(x)R_0\frac{\partial^4 Y(x,t)}{\partial x^2 \partial t^2} + G_K(x,t) = P(x,t) \tag{1}$$

The structure's changeable flexural stiffness is denoted by $EI(x)$, the time coordinate is denoted by t, the geographical coordinate by x, the transfer displacement is $Y(x,t)$, the variable mass per unit length of the non-uniform beam is $\mu(x)$. R_0 is the rotatory inertial factor, N is the constant axial force, the variable foundation reaction is $G_k(x,t)$, and the translating load is $P(x,t)$.

The foundation response and the lateral deflection $Y(x,t)$ have the following relationship:

$$G_K(x,t) = S(x)Y(x,t) - \frac{\partial}{\partial x}\left[k(x)\frac{\partial Y(x,t))}{\partial x}\right] \tag{2}$$

where the two variable parameters of the elastic foundation are $S(x)$ and $K(x)$, which stand for variable shear modulus and variable foundation stiffness, respectively.
When taking into account how the moving load affects the beam's reaction, the following is the shape of the load $P(x,t)$.

$$P(x,t) = P_f(x,t)\left[1 - \frac{d^2}{dt^2}\left[\frac{Y(x,t)}{g}\right]\right] \tag{3}$$

where the moving force $P_f(x,t)$ operating continuously on the beam model is expressed as

$$P_f(x,t) = Mg\cos\omega t H(x - f(t)) \tag{4}$$

that is

$$0 \leq f(t) \leq L \tag{5}$$

The Heaviside function, $H(x - f(t))$, is defined as

$$H(x - f(t)) = \begin{cases} 0, & x < f(t); \\ 1, & x \geq f(t). \end{cases} \tag{6}$$

The convective acceleration defined by [1] is $\frac{d^2}{dt^2}$, and g is the acceleration caused by gravity.

$$\frac{d^2}{dt^2} = \frac{\partial^2}{\partial t^2} + 2\frac{d}{dt}f(t)\frac{\partial^2}{\partial x \partial t} + \left(\frac{df(t)}{dt}\right)\frac{\partial^2}{\partial x^2} + \frac{d^2}{dt^2}f(t)\frac{\partial}{\partial x} \tag{7}$$

The distance traveled by the load at any given time is denoted by $f(t)$.

$$f(t) = x_o + ct + \frac{1}{2}at^2 \tag{8}$$

where c is the initial velocity, a is the constant acceleration of motion, and x_0 is the location of application of force $P(x,t)$ at the instance $t = 0$.

As an illustration, $S(x)$ and $K(x)$ in the problem [12] have the form

$$S(x) = S_0(4x - 3x^2 + x^3) \text{ and } K(x) = K_0(12 - 13x + 6x^2 + x^3) \tag{9}$$

S_0 is the foundation constant and K_0 is a constant shear Modulus.

Additionally, $I(x)$ and $\mu(x)$ are assumed to be of the form [13].

$$I(x) = I_0(1 + \sin\frac{\pi x}{L})^3 \text{ and } \mu(x) = \mu_o(1 + \sin\frac{\pi x}{L}) \tag{10}$$

using the equations (2)-(10) in (1), After simplications one obtains

$$\begin{aligned}
&\frac{EI(x)_0}{4}\left[\frac{\partial^2}{\partial x^2}\left(\left(10 + 15\sin\frac{\pi x}{L} - 6\cos\frac{2\pi x}{L} - \sin\frac{3\pi x}{L}\right)\frac{\partial^2 Y(x,t)}{\partial x^2}\right)\right] - N\frac{\partial^2 Y(x,t)}{\partial x^2} + \\
&\mu_0\left(1 + \sin\frac{\pi x}{L}\right)\frac{\partial^2 Y(x,t)}{\partial t^2} - \mu_0\left(1 + \sin\frac{\pi x}{L}\right)R_0\frac{\partial^4 Y(x,t)}{\partial x^2 \partial t^2} + S_0\left(4x - 3x^2 + x^3\right)Y(x,t) \\
&- K_0\left(-13 + 12x + 3x^2\right)\frac{\partial}{\partial x}Y(x,t) - K_0\left(12 - 13x + 6x^2 + x^3\right)\frac{\partial^2}{\partial x^2}Y(x,t) + \\
&M\cos\omega t H\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]\left[\frac{\partial^2}{\partial t^2} + 2(c + at)\frac{\partial^2}{\partial x \partial t} + (c + at)^2\frac{\partial^2}{\partial x^2} + a\frac{\partial}{\partial x}\right]Y(x,t) \\
&= Mg\cos\omega t H\left[x - \left(x_o + ct + \frac{1}{2}at^2\right)\right]
\end{aligned} \tag{11}$$

In this analysis, it is assumed that the non-uniform Rayleigh beam is simply supported. Hence, the following are the boundary conditions:

$$Y(0,t) = Y(L,t) = 0; \qquad \frac{\partial^2 Y(0,t)}{\partial x^2} = \frac{\partial^2 Y(L,t)}{\partial x^2} \tag{12}$$

In order to maintain generality, we assumed that the beam start from rest, with no initial deflection or velocity. the initial conditions is taken to be

$$Y(x,0) = 0 = \frac{\partial Y(x,0)}{\partial t} \tag{13}$$

Equation (11) represents a fourth order partial differential equation including variable coefficients for a non-uniform Rayleigh beam sitting on a non-uniform bi-parametric foundation and subjected to variable-magnitude accelerating masses. Along the length L of the beam, the beam's properties, such as its moment of inertia and mass per unit length, are thought to vary.

3 Approximate Solution

The Generalized Galerkin Method (GGM), defined in [12], is one of the approximation techniques most appropriate for addressing various issues in the dynamics of structures. According to this approach, the solution to equation (11) take the form:

$$Y_n(x,t) = \sum_{m=1}^{\infty} W_m(t) U_m(x) \tag{14}$$

where $U_m(x)$ is selected so as to satisfy the relevant elastic boundary condition. Substituting (14) into (11) and after some simplifications and arrangements, one obtains

$$\sum_{i=1}^{N} \Bigg[\Bigg(\Big(U_m(x) + \sin\frac{\pi x}{L}(U_m(x)) - R_0(U''_m(x) + \sin\frac{\pi x}{L} U''_m(x))\Big) \ddot{W}_m(t) + \Bigg(\frac{EI_0}{4\mu_0} \Bigg(10U^{iv}_m(x)$$
$$+ 15\sin\frac{\pi x}{L} U^{iv}_m(x) - 6\cos\frac{2\pi x}{L} U^{iv}_m(x) - \sin\frac{3\pi x}{L} U^{iv}_m(x) - \frac{30\pi}{L}\cos\frac{\pi x}{L} U'''_m(x) + \frac{24\pi}{L}\sin\frac{2\pi x}{L} U'''_m(x)$$
$$- \frac{6\pi}{L}\cos\frac{3\pi x}{L} U'''_m(x) - \frac{15\pi^2}{L^2}\sin\frac{\pi x}{L} U''_m(x) + \frac{24\pi^2}{L^2}\cos\frac{2\pi x}{L} U''_m(x) + \frac{9\pi^2}{L^2}\sin\frac{3\pi x}{L} U''_m(x) \Bigg) - \frac{N_0}{\mu_0} U''_m(x)$$
$$+ \frac{S_0}{\mu_0}\Big(4xU_m(x) - 3x^2U_m(x) + x^3U_m(x) \Big) - \frac{K_0}{\mu_0}\Big(-13U'_m(x) + 12xU'_m(x) + 3x^2U'_m(x) + 12U''_m(x)$$
$$- 13xU''_m(x) - 6x^2U''_m(x) + x^3U''_m(x) \Big) \Bigg] W_m(t) + \sum_{i=1}^{N} \frac{M\cos\omega t}{\mu_0} \Bigg(H\Big[x - \Big(x_o + ct + \frac{1}{2}at^2 \Big) \Big] U_m(x)\ddot{W}_m(t)$$
$$+ 2(c+at)H\Big[x - \Big(x_o + ct + \frac{1}{2}at^2 \Big) \Big] U'_m(x)\dot{W}_m(t) + (c+at)^2 H\Big[x - \Big(x_o + ct + \frac{1}{2}at^2 \Big) \Big] U''_m(x)W_m(t) +$$
$$aH\Big[x - \Big(x_o + ct + \frac{1}{2}at^2 \Big) \Big] U'_m(x)W_m(t) \Bigg) - \frac{Mg\cos\omega t}{\mu_0} H\Big[x - \Big(x_o + ct + \frac{1}{2}at^2 \Big) \Big] = 0 \tag{15}$$

where the first derivatives of $U_m(x)$ and $W_m(t)$ with respect to x and t, respectively, are $U'_m(x)$ and $\dot{W}_m(t)$. The expression on the right hand side of equation (15) must be orthogonal to function $U_k(x)$ in order to determine $W_m(t)$. Therefore, following arrangement and simplification, one gets

$$\sum_{m=0}^{N}\Bigg\{D_0(m,k)\ddot{W}_m(t)+D_1(m,k)W_m(t)+\frac{M\cos\omega t}{\mu_0}\Bigg(V_{25}(m,k)\ddot{W}_m(t)+2(c+at)V_{26}(m,k)\dot{W}_m(t)$$
$$+(c+at)V_{27}(m,k)W_m(t)+aV_{28}(m,k)W_m(t)\Bigg)\Bigg\}=\frac{Mg\cos\omega t}{\mu_0}V_{29}(t) \tag{16}$$

where

$$D_0(m.k)=(V_0(m,k)+V_1(m,k))-R_0(V_2(m,k)+V_3(m,k))$$

$$D_1(m,k)=\Bigg(Q_A\Bigg(10V_4(m,k)-15V_5(m,k)-6V_6(m,k)-V_7(m,k)+\frac{30\pi}{L}V_8(m,k)+24\frac{\pi}{L}V_9(m,k)$$
$$-6\frac{\pi}{L}V_{10}(m,k)-15\frac{\pi^2}{L^2}V_{11}(m,k)+24\frac{\pi^2}{L^2}V_{12}(m,k)+3\frac{\pi^2}{L^2}V_{13}(m,k)\Bigg)-Q_BV_{14}(m,k)$$
$$-Q_C\Bigg(4V_{15}(m,k)-3V_{16}(m,k)+V_{17}(m,k)\Bigg)-Q_D\Bigg(-13V_{18}(m,k)+12V_{19}(m,k)-3V_{20}(m,k)$$
$$+12V_{21}(m,k)-13V_{22}(m,k)-6V_{23}(m,k)-V_{24}(m,k)\Bigg)\Bigg)$$

$$Q_A=\frac{4I_0}{\mu_0},\quad Q_B=\frac{N_0}{\mu_0},\quad Q_C=\frac{S_0}{\mu_0}\quad and\quad Q_D=\frac{K_0}{\mu_0}$$

$$V_0(m,k)=\int_0^L U_m(x)U_k(x)dx$$

$$V_1(m,k)=\int_0^L \sin\frac{\pi x}{L}U_m(x)U_k(x)dx$$

$$V_2(m,k)=\int_0^L U''_m(x)U_k(x)dx$$

$$V_3(m,k)=\int_0^L \sin\frac{\pi x}{L}U''_m(x)U_k(x)dx$$

$$V_4(m,k)=\int_0^L U^{iv}_m(x)U_k(x)dx$$

$$V_5(m,k)=\int_0^L \sin\frac{\pi x}{L}U^{iv}_m(x)U_k(x)dx$$

$$V_6(m,k)=\int_0^L \cos\frac{2\pi x}{L}U^{iv}_m(x)U_k(x)dx$$

$$V_7(m,k)=\int_0^L \sin\frac{3\pi x}{L}U^{iv}_m(x)U_k(x)dx$$

$$
\begin{aligned}
V_8(m,k) &= \int_0^L \cos\frac{\pi x}{L} U'''_m(x)U_k(x)dx \\
V_9(m,k) &= \int_0^L \sin 2\frac{\pi x}{L} U'''_m(x)U_k(x)dx \\
V_{10}(m,k) &= \int_0^L \cos 3\frac{\pi x}{L} U'''_m(x)U_k(x)dx \\
V_{11}(m,k) &= \int_0^L \sin\frac{\pi x}{L} U''_m(x)U_k(x)dx \\
V_{12}(m,k) &= \int_0^L \cos 2\frac{\pi x}{L} U''_m(x)U_k(x)dx \\
V_{13}(m,k) &= \int_0^L \sin 3\frac{\pi x}{L} U''_m(x)U_k(x)dx \\
V_{14}(m,k) &= \int_0^L U_m(x)U_k(x)dx \\
V_{15}(m,k) &= \int_0^L xU_m(x)U_k(x)dx \\
V_{16}(m,k) &= \int_0^L x^2U_m(x)U_k(x)dx \\
V_{17}(m,k) &= \int_0^L x^3U_m(x)U_k(x)dx \\
V_{18}(m,k) &= \int_0^L U'_m(x)U_k(x)dx \\
V_{19}(m,k) &= \int_0^L xU'_m(x)U_k(x)dx \\
V_{20}(m,k) &= \int_0^L x^2U'_m(x)U_k(x)dx \\
V_{21}(m,k) &= B_2(m,k) \\
V_{22}(m,k) &= \int_0^L xU''_m(x)U_k(x)dx \\
V_{23}(m,k) &= \int_0^L x^2U''_m(x)U_k(x)dx \\
V_{24}(m,k) &= \int_0^L x^3U''_m(x)U_k(x)dx \\
V_{25}(m,k) &= \int_0^L H\left[x-\left(x_o+ct+\frac{1}{2}at^2\right)\right]U_m(x)U_k(x)dx \\
V_{26}(m,k) &= \int_0^L H\left[x-\left(x_o+ct+\frac{1}{2}at^2\right)\right]U'_m(x)U_k(x)dx \\
V_{27}(m,k) &= \int_0^L H\left[x-\left(x_o+ct+\frac{1}{2}at^2\right)\right]U''_m(x)U_k(x)dx \\
V_{28}(m,k) &= V_{26}(m,k) \\
V_{29}(m,k) &= \int_0^L H\left[x-\left(x_o+ct+\frac{1}{2}at^2\right)\right]U_k(x)dx
\end{aligned} \tag{18}
$$

With basic supports at edges $x = 0$ and $x = L$ in our elastic system, we select

$$U_m(x) = \frac{\sin m\pi x}{L} \text{ and } U_k(x) = \frac{\sin k\pi x}{L} \tag{19}$$

Equation (18) is solved by substituting expressions for $U_m(x)$ and $U_k(x)$, and by using the Heaviside unit step function's Fourier series representation, specifically;

$$H = \frac{1}{4} + \frac{1}{\pi}\sum_{0}^{\infty} \frac{\sin(2n+1)\pi(x - (x_0 + ct + \frac{1}{2}at^2))}{(2n+1)}, \qquad 0 < x < L \tag{20}$$

Several reductions in complexity and reorganizations gives

$$\begin{aligned}
&D_0(m,k)\ddot{W}_m(t) + D_1(m,k)W_m(t) + \Gamma_0 \cos\omega t\Bigg\{L\Bigg(\frac{1}{4}Q_1 + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{(2n+1)}Q_{1A} \\
&- \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{(2n+1)}Q_{1B}\Bigg)\ddot{W}_m(t) + 2L(c+at)\Bigg(\frac{1}{4}Q_2 \\
&+ \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{(2n+1)}Q_{2A} - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{(2n+1)}Q_{2B}\Bigg)\dot{W}_m(t) \\
&+ \Bigg(L(c+at)^2\Bigg(\frac{1}{4}Q_3 + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{(2n+1)}Q_{3A} \\
&- \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{(2n+1)}Q_{3B}\Bigg) + La\Bigg(\frac{1}{4}Q_2 \\
&+ \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{(2n+1)}Q_{2A} - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0 + ct + \frac{1}{2}at^2)}{(2n+1)}Q_{2B}\Bigg)\Bigg)W_m(t)\Bigg\} \\
&= \frac{LMg\cos\omega t}{k\pi\mu_0}\Bigg(-(-1)^k + \cos_{\frac{k\pi}{L}}(x_0 + ct + \frac{1}{2}at^2)\Bigg)
\end{aligned} \tag{21}$$

where

$$\begin{aligned}
Q_1 &= \int_0^L \sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}dx \\
Q_{1A} &= \int_0^L sin(2n+1)\pi x \sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}dx \\
Q_{1B} &= \int_0^L cos(2n+1)\pi x \sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}dx \\
Q_2 &= \frac{m\pi}{L}\int_0^L \cos\frac{m\pi x}{L}\sin\frac{k\pi x}{L}dx \\
Q_{2A} &= \frac{m\pi}{L}\int_0^L sin(2n+1)\pi x\cos\frac{m\pi x}{L}\sin\frac{k\pi x}{L}dx \\
Q_{2B} &= \frac{m\pi}{L}\int_0^L cos(2n+1)\pi x\cos\frac{m\pi x}{L}\sin\frac{k\pi x}{L}dx
\end{aligned}$$

$$Q_3 = -(\frac{m\pi}{L})^2 \int_0^L \sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}dx$$
$$Q_{3A} = -(\frac{m\pi}{L})^2 \int_0^L sin(2n+1)\pi x \sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}dx$$
$$Q_{3B} = -(\frac{m\pi}{L})^2 \int_0^L cos(2n+1)\pi x \sin\frac{m\pi x}{L}\sin\frac{k\pi x}{L}dx$$
$$\Gamma_0 = \frac{M}{\mu_0 L} \tag{22}$$

Γ_0 is the mass ratio

Equation (21) is the basic transformed equation of the non-uniform Rayleigh beam with simple support that is subjected to an accelerating force of varying size and rests on a bi-parametric foundation. We will address two instances of the equation in the sections that follow.

CASE I: Non-uniform Rayleigh beam Traversed by Moving Force

The classic situation of a moving force problem arises if we neglect the inertia factor. With a few simplifications and reorganizations, equation (21), under this supposition $\Gamma_0 = 0$, becomes

$$\ddot{W}_m(t) + \gamma_f^2 W_m(t) = P_m \cos\omega t\Big\{ - (-1)^k + \cos\frac{k\pi}{L}(x_0 + ct + \frac{1}{2}at^2)\Big\} \tag{23}$$

where.

$$P_m = \frac{LMg}{k\pi\mu_0 D_0(m,k)} \quad and \quad \gamma_f^2 = \frac{D_0(m,k)}{D_1(m,k)} \tag{24}$$

The approach of variation of parameters is used to solve equation (23). Firstly, it can be easily demonstrated that the homogeneous component of (23) has a generic solution that is provided by

$$W_c(t) = C_1\cos\gamma_f t + C_2\sin\gamma_f t \tag{25}$$

where C_1 and C_2 are constants. Thus a particular solution to equation (23) takes the form

$$W_p(t) = \tau_1(t)\cos\gamma_f t + \tau_2(t)\sin\gamma_f t \tag{26}$$

The functions that need to be determined are $\tau_1(t)$ and $\tau_2(t)$. It is easy to demonstrate from equation (23) that

$$\tau_1(t) = -\frac{-P_m}{\gamma_f}\Big\{\int \cos\omega t\Big(- (-1)^k + \cos\frac{k\pi}{L}(x_0 + ct + \frac{1}{2}at^2)\Big)\sin\gamma_f t\Big\}dt \tag{27}$$

$$\tau_2(t) = -\frac{-P_m}{\gamma_f}\Big\{\int \cos\omega t\Big(- (-1)^k + \cos\frac{k\pi}{L}(x_0 + ct + \frac{1}{2}at^2)\Big)\cos\gamma_f t\Big\}dt \tag{28}$$

By truncating the power series of sine and cosine at order 2, equations (27) and (28) becomes.

$$\tau_1(t) = -\frac{P_m}{2\gamma_f}\int\Big\{\theta_f(\sin\Theta_1 t + \sin\Theta_2 t) + (\sin\Theta_1 t + \sin\Theta_2 t)(\beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 + \beta_4 t^4)\Big\}dt \tag{29}$$

and,

$$\tau_2(t) = -\frac{P_m}{2\gamma_f}\int\Big\{\theta_f(\cos\Theta_1 t+\cos\Theta_2 t)+(\cos\Theta_1 t+\cos\Theta_2 t)(\beta_0+\beta_1 t+\beta_2 t^2+\beta_3+\beta_4 t^4)\Big\}dt \quad (30)$$

where

$$U_f = (\frac{K\pi}{L})^2, \quad \beta_0 = U_f xo^2, \quad \beta_1 = 2cU_f xo^2, \quad \beta_2 = U_f(axo^2+c^2), \quad \beta_3 = acU_f \quad \beta_4 = \frac{1}{4}a^2U_f$$
$$\theta_f = -(-1)^k, \quad \Theta_1 = \gamma_f+\omega, \quad \Theta_2 = \gamma_f-\omega \quad (31)$$

After simplification and arrangement, one obtains

$$\tau_1(t) = -\frac{P_n}{2\theta_f}\Big\{\theta_f(J_0+U_0)+(\beta_0J_0+\beta_1J_1+\beta_2J_2+\beta_3J_3+\beta_4J_4+\beta_0U_0+\beta_1U_1+\beta_2U_2+\beta_3U_3+\beta_4U_4)\Big\} \quad (32)$$

$$\tau_2(t) = -\frac{P_n}{2\theta_f}\Big\{\theta_f(V_0+\xi_0)+(\beta_0V_0+\beta_1V_1+\beta_2V_2+\beta_3V_3+\beta_4V_4+\beta_0\xi_0+\beta_1\xi_1+\beta_2\xi_2+\beta_3\xi_3+\beta_4\xi_4)\Big\} \quad (33)$$

where

$$J_0 = \int \sin\Theta_1 t dt \qquad V_0 = \int \cos\Theta_1 t dt$$
$$J_1 = \int t\sin\Theta_1 t dt \qquad V_1 = \int t\cos\Theta_1 t dt$$
$$J_2 = \int t^2\sin\Theta_1 t dt \qquad V_2 = \int t^2\cos\Theta_1 t dt$$
$$J_3 = \int t^3\sin\Theta_1 t dt \qquad V_3 = \int t^3\cos\Theta_1 t dt$$
$$J_4 = \int t^4\sin\Theta_1 t dt \qquad V_4 = \int t^4\cos\Theta_1 t dt$$
$$U_0 = \int \sin\Theta_2 t dt \qquad \xi_0 = \int \cos\Theta_2 t dt$$
$$U_1 = \int t\sin\Theta_2 t dt \qquad \xi_1 = \int t\cos\Theta_2 t dt$$
$$U_2 = \int t^2\sin\Theta_2 t dt \qquad \xi_2 = \int t^2\cos\Theta_2 t dt$$
$$U_3 = \int t^3\sin\Theta_2 t dt \qquad \xi_3 = \int t^3\cos\Theta_2 t dt$$
$$U_4 = \int t^4\sin\Theta_2 t dt \qquad \xi_4 = \int t^4\cos\Theta_2 t dt \quad (34)$$

solving the indefinate integrals in equation (34) and substitute into equations (32) and (33), one obtains

$$\tau_1(t) = \frac{-P_m}{2\gamma_f}\Big\{ -\Big(\theta_f + \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4\Big)\Big(\frac{\cos\Theta_1 t}{\Theta_1} + \frac{\cos\Theta_2 t}{\Theta_2}\Big)$$
$$+\Big(\beta_1 + 2\beta_2 t + 3\beta_3 t^2 + 4\beta_4 t^3\Big)\Big(\frac{\sin\Theta_1 t}{\Theta_1^2} + \frac{\sin\Theta_2 t}{\Theta_2^2}\Big) + \Big(2\beta_2 + 6\beta_3 t + 12\beta_4 t^2\Big)\Big(\frac{\cos\Theta_1 t}{\Theta_1^3} + \frac{\cos\Theta_2 t}{\Theta_2^3}\Big)$$
$$-\Big(6\beta_3 + 24\beta_4 t\Big)\Big(\frac{\sin\Theta_1 t}{\Theta_1^4} + \frac{\sin\Theta_2 t}{\Theta_2^4}\Big) - 24\beta_4\Big(\frac{\cos\Theta_1 t}{\Theta_1^5} + \frac{\cos\Theta_2 t}{\Theta_2^5}\Big)\Big\} \tag{35}$$

$$\tau_2(t) = \frac{-P_m}{2\gamma_f}\Big\{\Big(\theta_f + \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4\Big)\Big(\frac{\sin\Theta_1 t}{\Theta_1} + \frac{\sin\Theta_2 t}{\Theta_2}\Big)$$
$$+\Big(\beta_1 + 2\beta_2 t + 3\beta_3 t^2 + 4\beta_4 t^3\Big)\Big(\frac{\cos\Theta_1 t}{\Theta_1^2} + \frac{\cos\Theta_2 t}{\Theta_2^2}\Big) - \Big(2\beta_2 + 6\beta_3 t + 12\beta_4 t^2\Big)\Big(\frac{\sin\Theta_1 t}{\Theta_1^3} + \frac{\sin\Theta_2 t}{\Theta_2^3}\Big)$$
$$-\Big(6\beta_3 + 24\beta_4 t\Big)\Big(\frac{\cos\Theta_1 t}{\Theta_1^4} + \frac{\cos\Theta_2 t}{\Theta_2^4}\Big) + 24\beta_4\Big(\frac{\sin\Theta_1 t}{\Theta_1^5} + \frac{\sin\Theta_2 t}{\Theta_2^5}\Big)\Big\} \tag{36}$$

subtituting equations (35) and (36) into equation (26), the particular solution of the non-homogeneous second order differential equation takes the form

$$W_p(t) = \frac{-P_m}{2\gamma_f}\Big\{\Big(-\Big(\theta_f + \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4\Big)\Big(\frac{\cos\Theta_1 t}{\Theta_1} + \frac{\cos\Theta_2 t}{\Theta_2}\Big)$$
$$+\Big(\beta_1 + 2\beta_2 t + 3\beta_3 t^2 + 4\beta_4 t^3\Big)\Big(\frac{\sin\Theta_1 t}{\Theta_1^2} + \frac{\sin\Theta_2 t}{\Theta_2^2}\Big) + \Big(2\beta_2 + 6\beta_3 t + 12\beta_4 t^2\Big)\Big(\frac{\cos\Theta_1 t}{\Theta_1^3} + \frac{\cos\Theta_2 t}{\Theta_2^3}\Big)$$
$$-\Big(6\beta_3 + 24\beta_4 t\Big)\Big(\frac{\sin\Theta_1 t}{\Theta_1^4} + \frac{\sin\Theta_2 t}{\Theta_2^4}\Big) - 24\beta_4\Big(\frac{\cos\Theta_1 t}{\Theta_1^5} + \frac{\cos\Theta_2 t}{\Theta_2^5}\Big)\Big)\cos\gamma_f t$$
$$+\Big(\Big(\theta_f + \beta_0 + \beta_1 t + \beta_2 t^2 + \beta_3 t^3 + \beta_4 t^4\Big)\Big(\frac{\sin\Theta_1 t}{\Theta_1} + \frac{\sin\Theta_2 t}{\Theta_2}\Big)$$
$$+\Big(\beta_1 + 2\beta_2 t + 3\beta_3 t^2 + 4\beta_4 t^3\Big)\Big(\frac{\cos\Theta_1 t}{\Theta_1^2} + \frac{\cos\Theta_2 t}{\Theta_2^2}\Big) - \Big(2\beta_2 + 6\beta_3 t + 12\beta_4 t^2\Big)\Big(\frac{\sin\Theta_1 t}{\Theta_1^3} + \frac{\sin\Theta_2 t}{\Theta_2^3}\Big)$$
$$-\Big(6\beta_3 + 24\beta_4 t\Big)\Big(\frac{\cos\Theta_1 t}{\Theta_1^4} + \frac{\cos\Theta_2 t}{\Theta_2^4}\Big) + 24\beta_4\Big(\frac{\sin\Theta_1 t}{\Theta_1^5} + \frac{\sin\Theta_2 t}{\Theta_2^5}\Big)\Big)\sin\gamma_f t\Big\} \tag{37}$$

Consequently,

$$W_G(t) = W_c(t) + W_p(t) \tag{38}$$

Applying the initial conditional (13) to (38), the constants are found to be

$$C_1 = \frac{P_m}{2\gamma_f}\Big\{ -\Big(\theta_f + \beta_0\Big)\Big(\frac{1}{\Theta_1} + \frac{1}{\Theta_2}\Big) + 2\beta_2\Big(\frac{1}{\Theta_1^3} + \frac{1}{\Theta_2^3}\Big) - 24\beta_4\Big(\frac{1}{\Theta_1^5} + \frac{1}{\Theta_2^5}\Big)\Big\} \tag{39}$$

and

$$C_2 = \frac{P_m}{2\gamma_f}\Big\{\beta_1\Big(\frac{1}{\Theta_1^2} + \frac{1}{\Theta_2^2}\Big) - 6\beta_3\Big(\frac{1}{\Theta_1^4} + \frac{1}{\Theta_2^4}\Big)\Big\} \tag{40}$$

after certain simplifications and rearrangements, substituting (37), (39) and (40) into (38) and inverting the result yields

$$
\begin{aligned}
Y_m(x,t) = \sum_{n=0}^{\infty}\Bigg\{ -\frac{P_m}{2\gamma_f\Theta_1^5\Theta_2^5}\Bigg\{\Bigg(-\Big(\theta_f+\beta_0\Big)\Big(\Theta_1^4\Theta_2^5+\Theta_1^5\Theta_2^4\Big) + 2\beta_2\Big(\Theta_1^2\Theta_2^5+\Theta_1^5\Theta_2^2\Big) \\
- 24\beta_4\Big(\Theta_2^5+\Theta_1^5\Big)\Bigg)\cos\gamma_f t + \Bigg(\beta_1\Big(\Theta_1^3\Theta_2^5+\Theta_1^5\Theta_2^3\Big) - 6\beta_3\Big(\Theta_1\Theta_2^5+\Theta_1^5\Theta_2\Big)\Bigg)\sin\gamma_f t \\
+ \Bigg(\Bigg(-\Big(\theta_f+\beta_0+\beta_1 t+\beta_2 t^2+\beta_3 t^3+\beta_4 t^4\Big)\Big(\Theta_1^4\Theta_2^5\cos\Theta_1 t+\Theta_1^5\Theta_2^4\cos\Theta_2 t\Big) \\
+ \Big(\beta_1+2\beta_2 t+3\beta_3 t^2+4\beta_4 t^3\Big)\Big(\Theta_1^3\Theta_2^5\sin\Theta_1 t+\Theta_1^5\Theta_2^3\sin\Theta_2 t\Big) \\
+ \Big(2\beta_2+6\beta_3 t+12\beta_4 t^2\Big)\Big(\Theta_1^2\Theta_2^5\cos\Theta_1 t+\Theta_1^5\Theta_2^2\cos\Theta_2 t\Big) \\
- \Big(6\beta_3+24\beta_4 t\Big)\Big(\Theta_1\Theta_2^5\sin\Theta_1 t+\Theta_1^5\Theta_2\sin\Theta_2 t\Big) - 24\beta_4\Big(\Theta_2^5\cos\Theta_1 t+\Theta_1^5\cos\Theta_2 t\Big)\Bigg)\cos\gamma_f t \\
+ \Bigg(\Bigg(\Big(\theta_f+\beta_0+\beta_1 t+\beta_2 t^2+\beta_3 t^3+\beta_4 t^4\Big)\Big(\Theta_1^4\Theta_2^5\sin\Theta_1 t+\Theta_1^5\Theta_2^4\sin\Theta_2 t\Big) \\
+ \Big(\beta_1+2\beta_2 t+3\beta_3 t^2+4\beta_4 t^3\Big)\Big(\Theta_1^3\Theta_2^5\cos\Theta_1 t+\Theta_1^5\Theta_2^3\cos\Theta_2 t\Big) \\
- \Big(2\beta_2+6\beta_3 t+12\beta_4 t^2\Big)\Big(\Theta_1^2\Theta_2^5\sin\Theta_1 t+\Theta_1^5\Theta_2^2\sin\Theta_2 t\Big) \\
- \Big(6\beta_3+24\beta_4 t\Big)\Big(\Theta_1\Theta_2^5\cos\Theta_1 t+\Theta_1^5\Theta_2\cos\Theta_2 t\Big) \\
+ 24\beta_4\Big(\Theta_2^5\sin\Theta_1 t+\Theta_1^5\sin\Theta_2 t\Big)\Bigg)\sin\gamma_f t\Bigg)\Bigg\}\Bigg\} \times \sin\frac{m\pi}{L}x
\end{aligned} \tag{41}
$$

The transverse displacement response of a non-unifom simply supported Raleigh beam under varying magnitudes of accelerating loads while resting on a biparametric foundation is represented by equation (41).

CASE 2: Non-uniform Rayleigh beam Traversed by Moving Mass

The problem is known as the moving mass problem if the inertia term is kept in. In this instance, the full equation (21) must be solved. That is, if $\Gamma_0 \neq 0$.

It appears that due to the load magnitude fluctuation, the widely utilized Struble's asymptotic method was unable to solve the coupled second order ordinary differential equation. Consequently, we turn to the Runge-Kutta of fourth order approximate numerical solution method. Consequently, we rearrange equation (21)

$$
\begin{aligned}
D_0(m,k)\ddot{W}_m(t) + D_1(m,k)W_m(t) + \Gamma_0\cos\omega t\Big\{ Z_1(n,m,k)\ddot{W}_m(t) + 2L(c+at)Z_2(n,m,k)\dot{W}_m(t) \\
+ \Big(L(c+at)Z_3(n,m,k) + aLZ_2(n,m,k)\Big)W_m(t)\Big\} = \frac{LMg\cos\omega t}{k\pi\mu_0}\Big(-(-1)^k + \cos\frac{k\pi}{L}(x_0+ct+\frac{1}{2}at^2)\Big)
\end{aligned} \tag{42}
$$

where

$$Z_1(n,m,k) = \frac{1}{4}Q_1 + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{(2n+1)}Q_{1A} - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{(2n+1)}Q_{1B} \quad (43)$$

$$Z_2(n,m,k) = \frac{1}{4}Q_2 + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{(2n+1)}Q_{2A} - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{(2n+1)}Q_{2B} \quad (44)$$

$$Z_3(n,m,k) = \frac{1}{4}Q_3 + \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\cos(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{(2n+1)}Q_{3A} - \frac{1}{\pi}\sum_{n=0}^{\infty}\frac{\sin(2n+1)\pi(x_0+ct+\frac{1}{2}at^2)}{(2n+1)}Q_{3B} \quad (45)$$

futher simplification and arrangments of equation (42), one obtains

$$\ddot{W}_m(t) + DC_1\dot{W}_m(t) + DC_2W_m(t) = DC_3 \quad (46)$$

$$DC_1 = \frac{2L(c+at)\Gamma_0\cos\omega t Z_2(n,m,k)}{D_0(m,k)+\Gamma_0\cos\omega t Z_1(n,m,k)} \quad (47)$$

$$DC_2 = \frac{D_1(m,K)L\Gamma_0(C+at)^2\cos\omega t Z_3(n,m,k) + aL\Gamma_0\cos\omega t Z_2(n,m,k)}{D_0(m,k)+\Gamma_0\cos\omega t Z_1(n,m,k)} \quad (48)$$

$$DC_3 = \frac{LMg\cos\omega t}{\Big(D_0(m,k)+\Gamma_0\cos\omega t Z_1(n,m,k)\Big)\mu_0 k\pi}\Big(-(-1)^k + \cos\frac{k\pi}{L}(x_0+ct+\frac{1}{2}at^2)\Big) \quad (49)$$

The fourth order Runge-Kutta scheme is used to solve (49)

4 Numerical Results and Discussions

The non-uniform beam with length $L = 12.29m$, load velocity $c = 8.12m/s$, modulus of elasticity $E = 2.10924 \times 10^{-3}N/m^2$, and moment of inertia $I_0 = 2.87698 \times 10^9 kgm^2$ are used to demonstrate the analysis presented in this work.

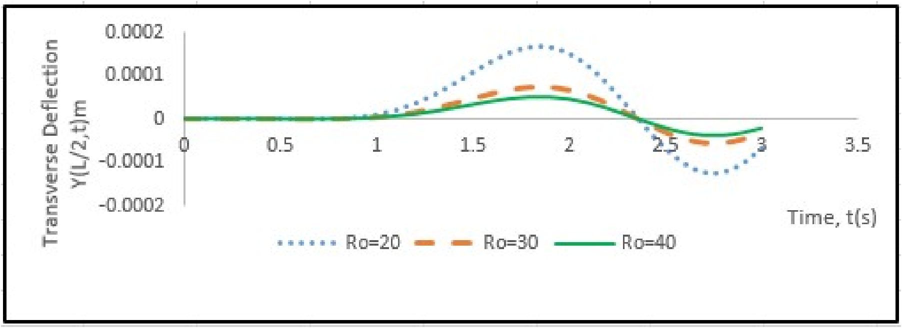

Fig. 1: Moving force deflection for a simply supported non-uniform Rayleigh beam under variable-magnitude accelerating masses and sitting on a bi-parametric foundation for fixed values of S_0, K_0, and N, for different amounts of rotatory inertia.

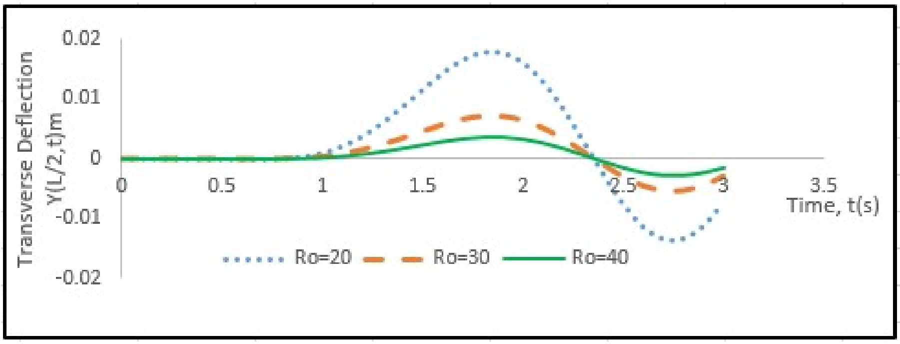

Fig. 2: Moving mass deflection for a simply supported non-uniform Rayleigh beam under variable-magnitude accelerating masses and sitting on a bi-parametric foundation for different values of the rotatory inertia, for fixed values of S_0, K_0, and N

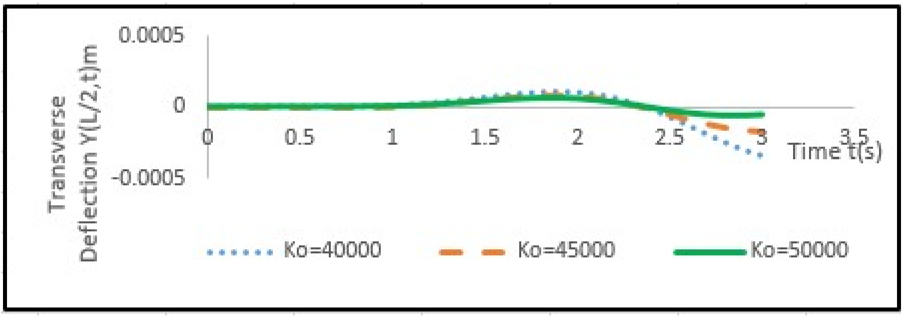

Fig. 3: Moving force deflection for a simply supported non-uniform Rayleigh beam under variable-magnitude accelerating masses and sitting on a bi-parametric foundation for fixed values of R_0, S_0 and N, for different amounts of Shear modulus.

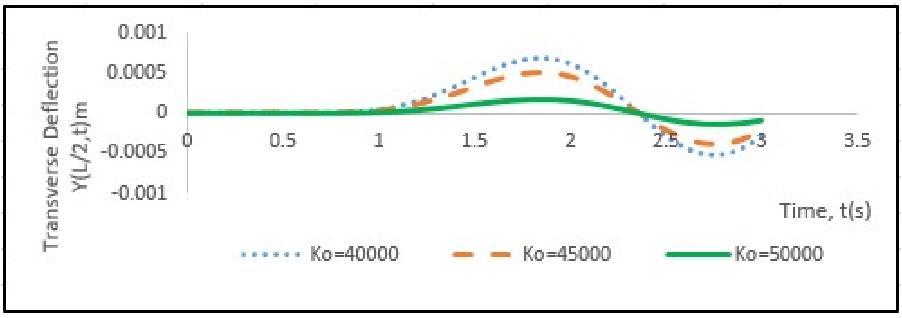

Fig. 4: Moving mass deflection for a simply supported non-uniform Rayleigh beam under variable-magnitude accelerating masses and sitting on a bi-parametric foundation for different values of the shear modulus, for fixed values of S_0, R_0 and N.

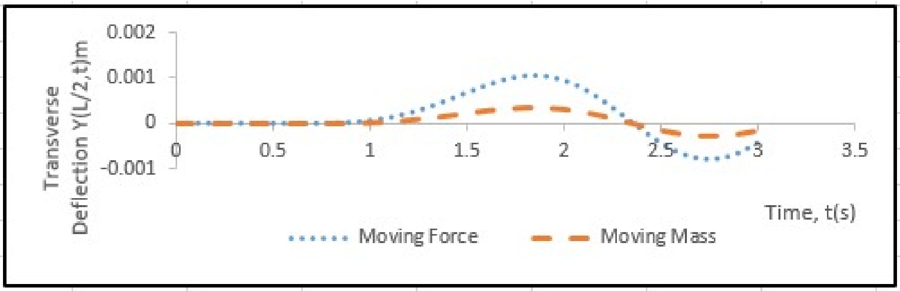

Fig. 5: Comparing the deflection profile of the moving mass and force of a simply supported non-uniform Rayleigh beam under variable-magnitude accelerating masses and sitting on a bi-parametric foundation with fixed values of K_0, N, R_0, and S_0.

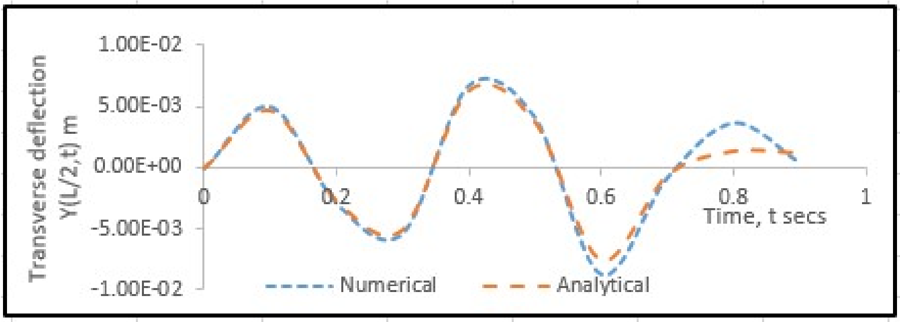

Fig. 6: Comparison of the displacement response of analytical solution and numerical solution of a simply supported non-uniform Rayleigh beam under variable-magnitude accelerating masses and sitting on a bi-parametric foundation for fixed values K_0, N, R_0, and S_0.

Fig. 1 and Fig. 2 shows the deflection profile of the simply supported non-uniform Rayleigh beam under variable magnitude acelerating masses for various values of rotatory inertia and fixed value of foundation stiffness S_0, shear modulus K_0 and axial force N for moving distributed force and moving distributed mass respectively . It is observed that the higher values of rotatory inertia reduce the deflection of the beam for both moving mass and moving force.

Similarly, Fig. 3 and Fig. 4 shows the deflection profile of the simply supported non-uniform Rayleigh beam under variable magnitude accelerating masses for various values of shear modulus and fixed value of foundation stiffness S_0, rotatory inertia R_0 and axial force N for moving distributed force and moving distributed mass respectively. It is observed that the higher values of shear modulus reduce the deflection of the beam for both moving mass and moving force.
Fig. 5 show the comparison of the moving mass and force of a simply supported non-uniform Rayleigh beam with fixed values of foundation stiffness S_0, rotatory inertia R_0, shear modulus K_0 and axial force N. The moving distributed force problem has a larger response amplitude than the moving distributed mass problem, and its critical speed is smaller than that of the moving distributed mass problem, according to the graphs. Therefore, in moving distributed forces as opposed to moving distributed masses, resonance is obtained earlier, highlighting the critical importance of considering both load types in beam design to ensure structural integrity and vibration control, ultimately enhancing bridge safety, building resilence and industrial machinery efficiency.
Finally, Fig 6. show the comparison between the numerical solution and approximate analytical solution for the deflection response of the moving distributed force of simply supported non-uniform Rayleigh beam resting on a bi-parametric foundation and under a variable-magnitude accelerating mass. Given that the amplitude of the two graph profiles is almost equal, it can be concluded that the Runge-Kutta method is a suitable approach for handling such a dynamic situation.

5 Conclusion

This paper presents a solution approach for the simply supported non-uniform Rayleigh beam resting on a bi-parametric foundation and under a variable-magnitude accelerating mass. The approximation procedure based on the generalized Galerkin method. The non-uniform Rayleigh beam's governing fourth-order differential equation with variable and singular coefficients is given closed-form solutions. Considerable attention is paid to the impact of relevant factors such the shear modulus, axial force, rotatory inertia correction factor, and foundation stiffness. Plotted curve analysis reveals that a decrease in the deflection of the simply supported non-uniform Rayleigh beam occurs with an increase in structural parameters. Bi-parametric foundations, thus, guarantee the safety of accelerating loads of varying magnitude while simultaneously reducing vibration.

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