Original Research Article

On Dual Generalized Guglielmo Numbers

Abstract. In this study, we investigate the generalized dual hyperbolic Guglielmo numbers and then various special cases are explored (including dual triangular numbers, dual triangular-Lucas numbers, dual oblong numbers, and dual pentagonal numbers). Binet's formulas, generating functions, and summation formulas for these numbers are presented. Additionally, Catalan's and Cassini's identities are provided, along with matrices associated with these sequences.

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1. Introduction

Dual numbers were first introduced by W.K. Clifford in 1873. This intriguing concept has numerous applications, including screw systems, modeling plane joints, iterative methods for displacement analysis of spatial mechanisms, inertial force analysis of spatial mechanisms, and more.

Here are some general information about the applications of dual numbers.

- Engineering and Physics:

  Used in electrical engineering and control systems.

  Applied in wave analysis and signal processing.

  Utilized in mechanical engineering for vibration analysis, among other applications.

- Mathematics and Geometry:

  Alongside complex numbers, dual numbers contribute to the extension of mathematical structures.

  Employed in geometry to represent various transformations.

- Computer Science:

Found in graphics and image processing.

Used in robotics and control systems for modeling and analysis.

- Finance and Economics:

  Applied in risk analysis and financial engineering.

  Utilized in option pricing and portfolio management.

- Optimization Problems:

  Used for finding solutions in optimization problems.

  Acts as a tool in linear programming and decision-making models.

- Quantum Mechanics:

  Employed in quantum computers and quantum mechanics for mathematical representation.

Next, we give some information raleted to hypercomplex number system and then we give some properities about dual number. As discussed in [15], the hypercomplex numbers systems are extensions of real numbers. Some examples of hypercomplex number systems ,which is commutative , are complex numbers, hyperbolic numbers and dual numbers.

- Complex numbers are formed by extending the real number system with the imaginary unit, denoted as ”i”, which satisfies the equation $i^2 = -1$. Complex numbers is defined as follows,

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\}.$$

- As discussed in [18], hyperbolic numbers extend the real number system with the hyperbolic unit j, where $j^2 = 1$. Hyperbolic numbers is defined as follows,

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}.$$

- As discussed in[10], dual numbers extend the real number system by introducing a new element ε, where $\varepsilon^2 = 0$. Dual numbers is defined as follows,

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

Let $\mathbb{D}$ =$\{d = a+\varepsilon b : a, b \in R, \varepsilon^2 = 0, \varepsilon \neq 0\} \subseteq \mathbb{R}\times\mathbb{R}$ is a set called dual numbers and we define following process on $\mathbb{D}$ for every $d_1 = x + x^*\varepsilon$, $d_2 = y + y^*\varepsilon \in \mathbb{D}$ as

$$\begin{aligned} + &: \mathbb{D}\times\mathbb{D} \to \mathbb{D},\ d_1 + d_2 = (x + x^*\varepsilon) + (y + y^*\varepsilon) = (x + y) + (x^* + y^*)\varepsilon, \\ \cdot &: \mathbb{D}\times\mathbb{D} \to \mathbb{D},\ d_1 \cdot d_2 = (x + x^*\varepsilon) \cdot (y + y^*\varepsilon) = xy + (xy^* + x^*y)\varepsilon, \\ d_1 &= (x + x^*\varepsilon) = (y + y^*\varepsilon) = d_2 \text{ if only if } x = x^*,\ y = y^*. \end{aligned}$$

Using above expressions we have following definations,

- – $(\mathbb{D}, +)$ is an abelian grup,

– $(\mathbb{D},+,\cdot)$ is commitative ring (where for every $d\in\mathbb{D}$ we have $d\cdot 1=d$ so that 1 is unit eleman on $\cdot$ process),
– $(\mathbb{D},+,\cdot)$ is not field because for every $d\in\mathbb{D}$ such that there is no element $d\cdot d'=d'\cdot d=1$,
– the $\mathbb{D}$ is a vector space on $\mathbb{R}$,
– $\widetilde{\mathbb{D}}=\{a+0\varepsilon : a\in\mathbb{R}\}$,which is subspace of $\mathbb{D}$, is isomorph $\mathbb{R}$,
– $(1,\varepsilon)$ is basis of $\mathbb{D}$,
– for every $d=(x+x^*\varepsilon)\in\mathbb{D}$ such that $\overline{d}=(x-x^*\varepsilon)\in\mathbb{D}$, $\frac{1}{d}=(\frac{1}{x}+\frac{x^*}{x}\varepsilon)\in\mathbb{D}$, $d\cdot\overline{d}=x^2$,$\overline{(\overline{d})}=d$
– for every $d_1=x+x^*\varepsilon$, $d_2=y+y^*\varepsilon\in\mathbb{D}$,$(y\neq 0)$, $\frac{d_1}{d_2}=(\frac{x}{y}+\frac{x^*-xy^*}{y^2}\varepsilon)\in\mathbb{D}$, $\overline{(\frac{d_1}{d_2})}=(\frac{\overline{d_1}}{\overline{d_2}})$, $\overline{(d_1+d_2)}=(\overline{d_1}+\overline{d_2})$ and $\overline{(d_1\cdot d_2)}=(\overline{d_1}\cdot\overline{d_2})$. For more detail see [25]

- Dual hyperbolic number is a type of hypercomplex number, specifically a member of the hyperbolic number system. A dual hyperbolic number is defined by

$$q=(a_0+ja_1)+\varepsilon(a_2+ja_3)=a_0+ja_1+\varepsilon a_2+\varepsilon ja_3$$

where $a_0,a_1,a_2,a_3\in\mathbb{R}$ are real numbers.

The set of all dual hyperbolic numbers are defined as

$$\mathbb{H}_{\mathbb{D}}=\{a_0+ja_1+\varepsilon a_2+\varepsilon ja_3 : a_0,a_1,a_2,a_3\in\mathbb{R},\ j^2=1, j\neq\pm 1,\varepsilon^2=0,\varepsilon\neq 0\}.$$

where ε denotes the pure dual unit $(\varepsilon^2=0,\varepsilon\neq 0)$, j denotes the hyperbolic unit $(j^2=1)$, and εj denotes the dual hyperbolic unit $((j\varepsilon)^2=0)$.

The $\{1,j,\varepsilon,\varepsilon j\}$ is linear independent and $\mathbb{H}_{\mathbb{D}}=sp\{1,j,\varepsilon,\varepsilon j\}$ so that $\{1,j,\varepsilon,\varepsilon j\}$ is a basis of $\mathbb{H}_{\mathbb{D}}$.For more detail see [3]. The next properties are holds for the base elements $\{1,j,\varepsilon,\varepsilon j\}$ of dual hyperbolic numbers (commutative multiplications):$1.\varepsilon=\varepsilon, 1.j=j$, $\varepsilon^2=\varepsilon.\varepsilon=(j\varepsilon)^2=0$, $j^2=j.j=1,\varepsilon.j=j.\varepsilon$, $\varepsilon.(\varepsilon j)=(\varepsilon j).\varepsilon=0$, $j(\varepsilon j)=(\varepsilon j)j=\varepsilon$.

Next, we will introduce a range of expressions associated with generalized Guglielmo numbers.

A generalized Guglielmo sequence, with the initial values W_0,W_1,W_2 not all being zero, $\{W_n\}_{n\geq 0}=\{W_n(W_0,W_1,W_2)\}_{n\geq 0}$ is defined by the third-order recurrence relations as follow

$$W_n=3W_{n-1}-3W_{n-2}+W_{n-3};\ W_0,W_1,W_2\quad(n\geq 2). \tag{1.1}$$

Therefore reccurance relation of $\{W_n\}_{n\geq 0}$ can be given to negative subscripts by defining

$$W_{-n}=3W_{-(n-1)}-3W_{-(n-2)}+W_{-(n-3)}$$

for $n=1,2,3,....$ As a result, recurrence (1.1) is true for all integer n.

In the Table 1 We provide the initial set of generalized Guglielmo numbers, both with positive and negative subscripts

Table 1. A few generalized Guglielmo numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$3W_0 - 3W_1 + W_2$
2	W_2	$6W_0 - 8W_1 + 3W_2$
3	$W_0 - 3W_1 + 3W_2$	$10W_0 - 15W_1 + 6W_2$
4	$3W_0 - 8W_1 + 6W_2$	$15W_0 - 24W_1 + 10W_2$
5	$6W_0 - 15W_1 + 10W_2$	$21W_0 - 35W_1 + 15W_2$
6	$10W_0 - 24W_1 + 15W_2$	$28W_0 - 48W_1 + 21W_2$

Throughout this paper we obtain W_n is the nth generalized Guglielmo numbers with the initial values W_0, W_1, W_2 where n is an integer.

When the initial values are $W_0 = 0, W_1 = 1, W_2 = 3$ we generate the triangular sequence, known as $\{T_n\}$, when the initial values are $W_0 = 3, W_1 = 3, W_2 = 3$ we generate the Triangular-Lucas sequence, known as $\{H_n\}$, when the initial values are $W_0 = 0, W_1 = 2, W_2 = 6$ we generate the oblong sequence $\{O_n\}$ and when the initial values are $W_0 = 0, W_1 = 1, W_2 = 5$ we generate the pentegonal sequence, known as $\{p_n\}$. In other words, triangular sequence $\{T_n\}_{n\geq 0}$, triangular-Lucas sequence $\{H_n\}_{n\geq 0}$, oblong sequence $\{O_n\}_{n\geq 0}$ and pentegonal sequence $\{p_n\}_{n\geq 0}$ are determined by the third-order recurrence relations

$$T_n = 3T_{n-1} - 3T_{n-2} + T_{n-3}, \qquad T_0 = 0, T_1 = 1, T_2 = 3, \tag{1.2}$$

$$H_n = 3H_{n-1} - 3H_{n-2} + H_{n-3}, \qquad H_0 = 3, H_1 = 3, H_2 = 3, \tag{1.3}$$

$$O_n = 3O_{n-1} - 3O_{n-2} + O_{n-3}, \qquad O_0 = 0, O_1 = 2, O_2 = 6, \tag{1.4}$$

$$p_n = 3p_{n-1} - 3p_{n-2} + p_{n-3}, \qquad p_0 = 0, p_1 = 1, p_2 = 5. \tag{1.5}$$

The sequences $\{T_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{O_n\}_{n\geq 0}$ and $\{p_n\}_{n\geq 0}$ can be extended to negative subscripts by defining,

$$\begin{aligned} T_{-n} &= 3T_{-(n-1)} - 3T_{-(n-2)} + T_{-(n-3)}, \\ H_{-n} &= 3H_{-(n-1)} - 3H_{-(n-2)} + H_{-(n-3)}, \\ O_{-n} &= 3O_{-(n-1)} - 3O_{-(n-2)} + O_{-(n-3)}, \\ p_{-n} &= 3p_{-(n-1)} - 3p_{-(n-2)} + p_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. As a result, recurrences (1.2)-(1.5) hold for all integer n.

We have the option to several essential properties of generalized Guglielmo numbers that are required.

- Binet formula of generalized Guglielmo sequence can be calculated using its characteristic equation given as

$$x^3 - 3x^2 + 3x - 1 = (x-1)^3 = 0.$$

The roots of the characteristic equation are given as follow

$$\alpha = \beta = \gamma = 1.$$

Binet formula are given, using these roots and the recurrence relation, as follow

$$W_n = A_1 + A_2 n + A_3 n^2 \tag{1.6}$$

where the coefficients of n above equality as

$$\begin{aligned} A_1 &= W_0, \\ A_2 &= \frac{1}{2}(-W_2 + 4W_1 - 3W_0), \\ A_3 &= \frac{1}{2}(W_2 - 2W_1 + W_0). \end{aligned} \tag{1.7}$$

Here, Binet formula of triangular, triangular-Lucas, oblong and pentagonal sequences are

$$\begin{aligned} T_n &= \frac{n(n+1)}{2}, \\ H_n &= 3, \\ O_n &= n(n+1), \\ p_n &= \frac{1}{2} n\left(3n-1\right). \end{aligned}$$

- The generating function of $\{W_n\} = \{W_n(W_0, W_1, W_2)\}$, for any integer n, is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 3W_0)x + (W_2 - 3W_1 + 3W_0)x^2}{1 - 3x + 3x^2 - x^3}. \tag{1.8}$$

- The Cassini identity for $\{W_n\} = \{W_n(W_0, W_1, W_2)\}$, for any integer n, is

$$W_{n+1}W_{n-1} - W_n^2 = -\frac{1}{2}\left(A + Bn + Cn^2\right) \tag{1.9}$$

where

$$\begin{aligned} A &= 2W_0^2 + 6W_1^2 - 6W_0W_1 - 2W_1W_2, \\ B &= -3W_0^2 - 8W_1^2 - W_2^2 + 10W_0W_1 - 4W_0W_2 + 6W_1W_2, \\ C &= W_0^2 + 4W_1^2 + W_2^2 - 4W_0W_1 + 2W_0W_2 - 4W_1W_2. \end{aligned}$$

If you require further information regarding generalized Guglielmo numbers, see [20]

Now, we give some information, related to dual ,hyperbolic, dual hyperbolic and other sequences, published in litarature.

- Cockle [8] studied the hyperbolic numbers with complex coefficients.
- Eren and Soykan [9] studied the generalized Generalized Woodall Numbers.
- Cheng and Thompson [6] introduced dual numbers with complex coefficients.
- Akar, Yüce and Şahin [3] presented the dual hyperbolic numbers.

- Soykan, Gümüş, Göcen [21] presented dual hyperbolic generalized Pell numbers given by

$$\widehat{V}_n = V_n + jV_{n+1} + \varepsilon V_{n+2} + j\varepsilon V_{n+3}$$

where generalized Pell numbers are given by $V_n = 2V_{n-1} + V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero.

- Cihan, Azak, Güngör, Tosun [2] studied dual hyperbolic Fibonacci and Lucas numbers given by, respectively,

$$\begin{aligned} DHF_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ DHL_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3} \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Soykan, Taşdemir and Okumuş [22] studied dual hyperbolic generalized Jacopsthal numbers given by

$$\widehat{J}_n = J_n + jJ_{n+1} + \varepsilon J_{n+2} + j\varepsilon J_{n+3}$$

where$J_n = J_{n-1} + 2J_{n-2}$, $J_0 = a$, $J_1 = b$.

- Bród, Liana, Włoch [5] studied dual hyperbolic generalized balancing numbers as

$$DHB_n = B_n + jB_{n+1} + \varepsilon B_{n+2} + j\varepsilon B_{n+3}$$

where $B_n = 6B_{n-1} - B_{n-2}$, $B_0 = 0$, $B_1 = 1$.

- Gürses, Şentürk, Yüce [11] studied dual-generalized complex Fibonacci and Lucas numbers, respectively, as

$$\begin{aligned} \widetilde{\mathcal{F}}_n &= F_n + jF_{n+1} + \varepsilon F_{n+2} + j\varepsilon F_{n+3}, \\ \widetilde{\mathcal{L}}_n &= L_n + jL_{n+1} + \varepsilon L_{n+2} + j\varepsilon L_{n+3}, \end{aligned}$$

where Fibonacci and Lucas numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$, $L_n = L_{n-1} + L_{n-2}$, $L_0 = 2$, $L_1 = 1$.

- Nurkan ,Guven, [17] studied Dual Fibonacci Quaternions as

$$\widetilde{Q}n = (F_n + F_{n+1}) + i(F_{n+1} + F_{n+2}) + j(F_{n+2} + F_{n+3}) + k(F_{n+3} + F_{n+4})$$

where Fibonacci given by $F_n = F_{n-1} + F_{n-2}$, $F_0 = 0$, $F_1 = 1$.

- Aydın [1] studied Dual Jacobsthal Quaternions as

$$QJ_{k;n} = J_{k;n} + i_1 J_{k;n+1} + i_2 J_{k;n+2} + i_3 J_{k;n+3}$$

where $J_n = J_{n-1} + 2J_{n-2},\ J_0 = 0,\ J_1 = 1.$

- Halici [13] studied Dual Fibonacci Octonions as

$$p = \sum_{s=0}^{7} F_{n+s} e_s$$

where Fibonacci given by $F_n = F_{n-1} + F_{n-2},\ F_0 = 0,\ F_1 = 1.$

Next section, we present the dual hyperbolic generalized Guglielmo numbers and give some properties of these numbers.

2. Dual Generalized Guglielmo Numbers and their Generating Functions and Binet's Formulas

In this section, we define dual generalized Guglielmo numbers then we present generating functions and Binet formulas for these numbers.

On the set of $\mathbb{H}_{\mathbb{D}}$, we will now explore dual generalized Guglielmo numbers on $\mathbb{H}$.The nth generalized dual Guglielmo numbers, with $\widetilde{W}_0, \widetilde{W}_1, \widetilde{W}_2$ being the initial conditions, are defined as follows

$$\widetilde{W}_n = W_n + \varepsilon W_{n+1}. \tag{2.1}$$

in addition (2.1) can be written to negative subscripts by defining,

$$\widetilde{W}_{-n} = W_{-n} + \varepsilon W_{-n+1} \tag{2.2}$$

so identity (2.1) holds for all integers n.

Now we define some special cases of dual generalized Guglielmo numbers. The nth dual triangular numbers, the nth dual triangular-Lucas numbers, the nth dual oblong numbers and the nth dual pentegonal numbers, respectively, are given as

the nth generalized dual triangular numbers $\widetilde{T}_n = T_n + \varepsilon T_{n+1}$, with $\widetilde{T}_0, \widetilde{T}_1, \widetilde{T}_2$ being the initial conditions, are defined as follows

$$\widetilde{T}_n = T_n + \varepsilon T_{n+1}$$

where

$$\widetilde{T}_0 = T_0 + \varepsilon T_1, \widetilde{T}_1 = T_1 + \varepsilon T_2, \widetilde{T}_2 = T_2 + \varepsilon T_3,$$

the nth generalized dual triangular-Lucas numbers $\widetilde{H}_n = H_n + \varepsilon H_{n+1}$, with $\widetilde{H}_0, \widetilde{H}_1, \widetilde{H}_2$ being the initial conditions, are defined as follows

$$\widetilde{H}_n = \ H_n + j\ H_{n+1}$$

where

$$\widetilde{H}_0 = \ H_0 + \varepsilon\ H_1, \widetilde{H}_1 = \ H_1 + \varepsilon\ H_2, \widetilde{H}_2 = \ H_2 + \varepsilon\ H_3,$$

the nth generalized dual triangular numbers $\widetilde{O}_n = O_n + \varepsilon O_{n+1}$, with $\widetilde{O}_0, \widetilde{O}_1, \widetilde{O}_2$ being the initial conditions, are defined as follows

$$\widetilde{O}_n = O_n + \varepsilon O_{n+1}$$

where

$$\widetilde{O}_0 = O_0 + \varepsilon O_1, \widetilde{O}_1 = O_1 + \varepsilon O_2, \widetilde{O}_2 = O_2 + \varepsilon O_3,$$

the nth generalized dual triangular numbers $\widetilde{p}_n = p_n + j p_{n+1}$, with $\widetilde{p}_0, \widetilde{p}_1, \widetilde{p}_2$ being the initial conditions, are defined as follows

$$\widetilde{p}_n = p_n + \varepsilon p_{n+1}$$

where

$$\widetilde{p}_0 = p_0 + \varepsilon p_1, \widetilde{p}_1 = p_1 + \varepsilon p_2, \widetilde{p}_2 = p_2 + \varepsilon p_3.$$

For dual triangular numbers, taking $W_n = T_n$, $T_0 = 0$, $T_1 = 1$, $T_2 = 3$, we get

$$\widetilde{T}_0 = 3\varepsilon, \widetilde{T}_1 = 1 + 6\varepsilon, \widetilde{T}_2 = 3 + 10\varepsilon,$$

for dual triangular-Lucas numbers, taking $W_n = H_n$, $H_0 = 3$, $H_1 = 3$, $H_2 = 3$, we get

$$\widetilde{H}_0 = 3 + 3\varepsilon, \widetilde{H}_1 = 3 + 3\varepsilon, \widetilde{H}_2 = 3 + 3\varepsilon,$$

for dual oblong numbers,taking $W_n = O_n$, $O_0 = 0$, $O_1 = 2$, $O_2 = 6$, we get

$$\widetilde{O}_0 = 6\varepsilon, \widetilde{O}_1 = 2 + 12\varepsilon, \widetilde{O}_2 = 6 + 20\varepsilon,$$

and for dual pentegonal numbers, taking $W_n = p_n$, $p_0 = 0$, $p_1 = 1$, $p_2 = 5$, we get

$$\widetilde{p}_0 = 5\varepsilon, \widetilde{p}_1 = 1 + 12\varepsilon, \widetilde{p}_2 = 5 + 22\varepsilon,$$

Thus, by using (2.1), we can formulate the following identity for non-negative integers n,

$$\widetilde{W}_n = 3\widetilde{W}_{n-1} - 3\widetilde{W}_{n-2} + \widetilde{W}_{n-3}. \tag{2.3}$$

Hence the sequence $\{\widetilde{W}_n\}_{n\geq 0}$ can be given as

$$\widetilde{W}_{-n} = 3\widetilde{W}_{-(n-1)} - 3\widetilde{W}_{-(n-2)} + \widetilde{W}_{-(n-3)},$$

for $n \in \{1, 2, 3....\}$ by using (2.2). Accordingly, recurrence (2.3) is true for all integer n.

In the Table 2, We provide the initial dual generalized Guglielmo numbers with both positive and negative subscripts.

Table 2. Some dual generalized Guglielmo numbers

n	$\widetilde{W}_n$	$\widetilde{W}_{-n}$
0	$\widetilde{W}_0$	$\widetilde{W}_0$
1	$\widetilde{W}_1$	$3\widetilde{W}_0 - 3\widetilde{W}_1 + \widetilde{W}_2$
2	$\widetilde{W}_2$	$6\widetilde{W}_0 - 8\widetilde{W}_1 + 3\widetilde{W}_2$
3	$\widetilde{W}_0 - 3\widetilde{W}_1 + 3\widetilde{W}_2$	$10\widetilde{W}_0 - 15\widetilde{W}_1 + 6\widetilde{W}_2$
4	$3\widetilde{W}_0 - 8\widetilde{W}_1 + 6\widetilde{W}_2$	$15\widetilde{W}_0 - 24\widetilde{W}_1 + 10\widetilde{W}_2$
5	$6\widetilde{W}_0 - 15\widetilde{W}_1 + 10\widetilde{W}_2$	$21\widetilde{W}_0 - 35\widetilde{W}_1 + 15\widetilde{W}_2$
6	$10\widetilde{W}_0 - 24\widetilde{W}_1 + 15\widetilde{W}_2$	$28\widetilde{W}_0 - 48\widetilde{W}_1 + 21\widetilde{W}_2$

Note that

$$
\begin{aligned}
\widetilde{W}_0 &= W_0 + \varepsilon W_1, \\
\widetilde{W}_1 &= W_1 + \varepsilon W_2, \\
\widetilde{W}_2 &= W_2 + \varepsilon W_3.
\end{aligned}
$$

Some dual triangular numbers, dual triangular-Lucas numbers, dual oblong numbers, and dual pentagonal numbers with positive or negative subscripts are presented tables which is given below .

Table 3. dual triangular numbers

n	$\widetilde{T}_n$	$\widetilde{T}_{-n}$
0	ε	
1	$1+3\varepsilon$	0
2	$3+6\varepsilon$	1
3	$6+10\varepsilon$	$3+\varepsilon$
4	$10+15\varepsilon$	$6+3\varepsilon$
5	$15+21\varepsilon$	$10+6\varepsilon$

Table 4. dual triangular-Lucas numbers

n	$\widetilde{H}_n$	$\widetilde{H}_{-n}$
0	$3+3\varepsilon$	
1	$3+3\varepsilon$	$3+3\varepsilon$
2	$3+3\varepsilon$	$3+3\varepsilon$
3	$3+3\varepsilon$	$3+3\varepsilon$
4	$3+3\varepsilon$	$3+3\varepsilon$
5	$3+3\varepsilon$	$3+3\varepsilon$

Table 5. dual oblong numbers

n	$\widetilde{O}_n$	$\widetilde{O}_{-n}$
0	2ε	
1	$2+6\varepsilon$	
2	$6+12\varepsilon$	2
3	$12+20\varepsilon$	$6+2\varepsilon$
4	$20+30\varepsilon$	$12+6\varepsilon$
5	$30+42\varepsilon$	$20+12\varepsilon$

Table 6. dual pentegonal numbers

n	$\widetilde{p}_n$	$\widetilde{p}_{-n}$
0	ε	
1	$1+5\varepsilon$	2
2	$5+12\varepsilon$	$7+2\varepsilon$
3	$12+22\varepsilon$	$15+7\varepsilon$
4	$22+35\varepsilon$	$26+15\varepsilon$
5	$35+51\varepsilon$	$40+26\varepsilon$

Now, we will establish Binet's formula for the dual generalized Guglielmo numbers, and for the remainder of the study, we will utilize the following notations:

$$\widetilde{\alpha} = 1 + \varepsilon, \tag{2.4}$$

$$\widetilde{\beta} = \varepsilon. \tag{2.5}$$

Note that the following identities are true:

$$\begin{aligned} \widetilde{\alpha}^2 &= 1 + 2\varepsilon, \\ \widetilde{\beta}^2 &= 0, \\ \widetilde{\alpha}\widetilde{\beta} &= \widetilde{\beta}. \end{aligned}$$

THEOREM 1. *(Binet's Formula) For any integer n, the nth dual generalized Guglielmo number can be expressed as follows*

$$\widetilde{W}_n = (\widetilde{\alpha}A_1 + \widetilde{\beta}(A_2 + A_3)) + (\widetilde{a}A_2 + 2\widetilde{\beta}A_3)n + \widetilde{a}A_3 n^2 \tag{2.6}$$

where $\widetilde{\alpha}$, $\widetilde{\beta}$ are given as (2.4)-(2.5)

Proof. Using (1.6) and (1.7)) we can write following identity

$$\begin{aligned} \widetilde{W}_n &= W_n + \varepsilon W_{n+1}, \\ &= A_1 + A_2 n + A_3 n^2 + (A_1 + A_2 (n+1) + A_3 (n+1)^2)\varepsilon \\ &= (\widetilde{\alpha}A_1 + \widetilde{\beta}(A_2 + A_3)) + (\widetilde{a}A_2 + 2\widetilde{\beta}A_3)n + \widetilde{a}A_3 n^2. \end{aligned}$$

This proves (2.6). □

As special cases, for any integer n, the Binet's Formula of nth dual triangual numbers, the Binet's Formula of nth dual triangular-Lucas numbers, the Binet's Formula of nth dual oblong numbers and the Binet's Formula of nth dual pentegonal numbers, respectively, are

$$\begin{aligned} \widetilde{T}_n &= \frac{1}{2}(\widetilde{\beta} + (\widetilde{\alpha} + 2\widetilde{\beta})n + \widetilde{\alpha}n^2), \\ \widehat{H}_n &= 3\widetilde{\alpha}, \\ \widetilde{O}_n &= 2\widetilde{\beta} + \left(\widetilde{\alpha} + 2\widetilde{\beta}\right) n + \widetilde{\alpha}n^2, \\ \widetilde{p}_n &= \frac{1}{2}(2\widetilde{\beta} + \left(6\widetilde{\beta} - \widetilde{\alpha}\right) n + 3\widetilde{\alpha}n^2). \end{aligned}$$

Next, we will obtain the generating function of the dual generalized Guglielmo numbers.

THEOREM 2. *The generating function for the dual generalized Guglielmo numbers is*

$$f_{\widetilde{W}_n}(x) = \frac{\widetilde{W}_0 + (\widetilde{W}_1 - 3\widetilde{W}_0)x + (\widetilde{W}_2 - 3\widetilde{W}_1 + 3\widetilde{W}_0)x^2}{(1 - 3x + 3x^2 - x^3)}. \tag{2.7}$$

Proof. Let the generating function of the dual generalized Guglielmo numbers is given below

$$f_{\widetilde{W}_n}(x) = \sum_{n=0}^{\infty} \widetilde{W}_n x^n.$$

Following that, by utilizing the definition of the dual generalized Guglielmo numbers, and substracting $xg(x)$ and $x^2g(x)$ from $g(x)$, we get

$$\begin{aligned}
(1-3x+3x^2-x^3)f_{G\widetilde{W}_n}(x) &= \sum_{n=0}^{\infty}\widetilde{W}_n x^n - 3x\sum_{n=0}^{\infty}\widetilde{W}_n x^n + 3x^2\sum_{n=0}^{\infty}\widetilde{W}_n x^n - x^3\sum_{n=0}^{\infty}\widetilde{W}_n x^n, \\
&= \sum_{n=0}^{\infty}\widetilde{W}_n x^n - 3\sum_{n=0}^{\infty}\widetilde{W}_n x^{n+1} + 3\sum_{n=0}^{\infty}\widetilde{W}_n x^{n+2} - \sum_{n=0}^{\infty}\widetilde{W}_n x^{n+3}, \\
&= \sum_{n=0}^{\infty}\widetilde{W}_n x^n - 3\sum_{n=1}^{\infty}\widetilde{W}_{n-1} x^{n} + 3\sum_{n=2}^{\infty}\widetilde{W}_{n-2} x^{n} - \sum_{n=3}^{\infty}\widetilde{W}_{n-3} x^{n}, \\
&= (\widetilde{W}_0 + \widetilde{W}_1 x + \widetilde{W}_2 x^2) - 3(\widetilde{W}x + \widetilde{W}_1 x^2) + 3GW_0 x^2 \\
&\quad + \sum_{n=3}^{\infty}(\widetilde{W}_n - 3\widetilde{W}_{n-1} + 3\widetilde{W}_{n-2} - \widetilde{W}_{n-3})x^n, \\
&= \widetilde{W}_0 + \widetilde{W}_1 x + \widetilde{W}_2 x^2 - 3\widetilde{W}_0 x - 3\widetilde{W}_1 x^2 + 3\widetilde{W}_0 x^2, \\
&= \widetilde{W}_0 + (\widetilde{W}_1 - 3\widetilde{W}_0)x + (\widetilde{W}_2 - 3\widetilde{W}_1 + 3\widetilde{W}_0)x^2.
\end{aligned}$$

Note that we use the recurrence relation $\widetilde{W}_n = 3\widetilde{W}_{n-1} - 3\widetilde{W}_{n-2} + \widetilde{W}_{n-3}$. We rearrange equation which is given above then we obtain (2.7). □

As specific cases, the generating functions of the dual triangular, triangular-Lucas, oblong and dual pentegonal numbers are given by

$$\begin{aligned}
f_{\widetilde{T}_n}(x) &= \frac{(j+3\varepsilon+6j\varepsilon)+(1-8j\varepsilon-3\varepsilon)\,x+(\varepsilon+3j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \\
f_{\widetilde{H}_n}(x) &= \frac{(3+3j+3\varepsilon+3j\varepsilon)+(-6-6j-6\varepsilon-6j\varepsilon)\,x+(3+3j+3\varepsilon+3j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \\
f_{\widetilde{O}_n}(x) &= \frac{(2j+6\varepsilon+12j\varepsilon)+(2-16j\varepsilon-6\varepsilon)\,x+(2\varepsilon+6j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)}, \\
f_{\widetilde{p}_n}(x) &= \frac{(j+5\varepsilon+12j\varepsilon)+(1+2j-3\varepsilon-14j\varepsilon)\,x+(2+\varepsilon+5j\varepsilon)\,x^2}{(1-3x+3x^2-x^3)},
\end{aligned}$$

respectively. □

3. Deriving Binet's formula from the generating function

Next, we will explore the Binet's formula for the dual generalized Guglielmo numbers $\{\widetilde{W}_n\}$ by utilizing generating function $f_{\widetilde{W}_n}(x)$.

THEOREM 3. *(Binet formula of dual generalized Guglielmo numbers)*

(3.1) $$\widetilde{W}_n = (\widetilde{\alpha}A_1 + \widetilde{\beta}(A_2 + A_3)) + (\widetilde{a}A_2 + 2\widetilde{\beta}A_3)n + \widetilde{a}A_3 n^2.$$

Proof. We write

(3.2) $$\sum_{n=0}^{\infty}\widetilde{W}_n x^n = \frac{\widetilde{W}_0 + (\widetilde{W}_1 - 3\widetilde{W}_0)x + (\widetilde{W}_2 - 3\widetilde{W}_1 + 3\widetilde{W}_0)x^2}{(1-3x+3x^2-x^3)} = \frac{d_1}{(1-x)} + \frac{d_2}{(1-x)^2} + \frac{d_3}{(1-x)^3},$$

so that

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{W}_n x^n &= \frac{d_1}{(1-x)}+\frac{d_2}{(1-x)^2}+\frac{d_3}{(1-x)^3},\\ &= \frac{d_1(1-x)^2+d_2(1-x)+d_3}{(1-x)^3},\end{aligned}$$

then, we get

$$\widetilde{W}_0+(\widetilde{W}_1-3\widetilde{W}_0)x+(\widetilde{W}_2-3\widetilde{W}_1+3\widetilde{W}_0)x^2=(d_1+d_2+d_3)+(-2d_1-d_2)x+d_1x^2.$$

Ensuring equality of coefficients for the terms x of the same degree, we obtain

$$\begin{aligned}\widetilde{W}_0 &= d_1+d_2+d_3,\\ \widetilde{W}_1-3\widetilde{W}_0 &= -2d_1-d_2,\\ \widetilde{W}_2-3\widetilde{W}_1+3\widetilde{W}_0 &= d_1.\end{aligned} \tag{3.3}$$

Solving the (3.3), we can derive the following identities

$$\begin{aligned}d_1 &= 3\widetilde{W}_0-3\widetilde{W}_1+\widetilde{W}_2,\\ d_2 &= 5\widetilde{W}_1-3\widetilde{W}_0-2\widetilde{W}_2,\\ d_3 &= \widetilde{W}_0-2\widetilde{W}_1+\widetilde{W}_2.\end{aligned}$$

Thus (3.2) stated as follows

$$\begin{aligned}\sum_{n=0}^{\infty}\widetilde{W}_n x^n &= d_1\sum_{n=0}^{\infty}x^n+d_2\sum_{n=0}^{\infty}(n+1)x^n+d_3\sum_{n=0}^{\infty}\frac{n^2+3n+2}{2}x^n,\\ &= \sum_{n=0}^{\infty}(d_1+d_2(n+1)+d_3\frac{n^2+3n+2}{2})x^n,\\ &= \sum_{n=0}^{\infty}(\widetilde{W}_0+\frac{1}{2}(-\widetilde{W}_2+4\widetilde{W}_1-3\widetilde{W}_0)n+\frac{1}{2}(\widetilde{W}_2-2\widetilde{W}_1+\widetilde{W}_0)n^2)x^n.\end{aligned}$$

Consequently, we get

$$\widetilde{W}_n=\widetilde{A}_1+\widetilde{A}_2n+\widetilde{A}_3n^2$$

where

$$\begin{aligned}\widetilde{A}_1 &= \widetilde{W}_0,\\ \widetilde{A}_2 &= \frac{1}{2}(-\widetilde{W}_2+4\widetilde{W}_1-3\widetilde{W}_0),\\ \widetilde{A}_3 &= \frac{1}{2}(\widetilde{W}_2-2\widetilde{W}_1+\widetilde{W}_0).\end{aligned}$$

Take note that the following equalities are valid.

$$
\begin{aligned}
\widetilde{A}_1 &= \widetilde{W}_0 \\
&= W_0 + \varepsilon W_1 \\
&= (1+\varepsilon)W_0 + \varepsilon(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)) + (\varepsilon)(\frac{1}{2}(W_2 - 2W_1 + W_0)) \\
&= \widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3,
\end{aligned} \tag{3.4}
$$

$$
\begin{aligned}
\widetilde{A}_2 &= \frac{1}{2}(-\widetilde{W}_2 + 4\widetilde{W}_1 - 3\widetilde{W}_0) \\
&= \frac{1}{2}((-3W_0 + 4W_1 - W_2) + \varepsilon(-W_0 + W_2) \\
&= (1+\varepsilon)(\frac{1}{2}(-W_2 + 4W_1 - 3W_0)) + \varepsilon((W_2 - 2W_1 + W_0)) \\
&= (\widehat{a}A_2 + 2\widehat{\beta}A_3),
\end{aligned} \tag{3.5}
$$

$$
\begin{aligned}
\widetilde{A}_3 &= \frac{1}{2}(\widetilde{W}_2 - 2\widetilde{W}_1 + \widetilde{W}_0) \\
&= \frac{1}{2}((W_2 - 2W_1 + W_0) + \varepsilon(W_2 - 2W_1 + W_0) \\
&= \widetilde{a}A_3.
\end{aligned} \tag{3.6}
$$

The following equality can be written by using (3.4), (3.5) and (3.6).

$$\widetilde{W}_n = (\widehat{\alpha}A_1 + \widehat{\beta}A_2 + \widehat{\gamma}A_3) + (\widehat{a}A_2 + 2\widehat{\beta}A_3)n + \widehat{a}A_3n^2. \ \square$$

4. Some Identities Related to Dual Generalized Guglielmo numbers

We will now introduce some specific identities, i.e Simpson's formula, Catalan's identity and Cassini's identity, for the dual generalized Guglielmo sequence $\{\widetilde{W}_n\}$. The next theorem gives the Simpson's formula for the dual generalized Guglielmo numbers.

THEOREM 4. *(Simpson's formula for dual generalized Guglielmo numbers) For all integers n we have,*

$$
\begin{vmatrix} \widetilde{W}_{n+2} & \widetilde{W}_{n+1} & \widetilde{W}_n \\ \widetilde{W}_{n+1} & \widetilde{W}_n & \widetilde{W}_{n-1} \\ \widetilde{W}_n & \widetilde{W}_{n-1} & \widetilde{W}_{n-2} \end{vmatrix} = \begin{vmatrix} \widetilde{W}_2 & \widetilde{W}_1 & \widetilde{W}_0 \\ \widetilde{W}_1 & \widetilde{W}_0 & \widetilde{W}_{-1} \\ \widetilde{W}_0 & \widetilde{W}_{-1} & \widetilde{W}_{-2} \end{vmatrix}. \tag{4.1}
$$

Proof. First we assume that $n \geqslant 0$. For the proof, we employ mathematical induction on n. For $n = 0$ identity (4.1) is true. Now we take (4.1) is true for $n = k$. Therfore, the following identity can be written

$$
\begin{vmatrix} \widetilde{W}_{k+2} & \widetilde{W}_{k+1} & \widetilde{W}_k \\ \widetilde{W}_{k+1} & \widetilde{W}_k & \widetilde{W}_{k-1} \\ \widetilde{W}_k & \widetilde{W}_{k-1} & \widetilde{W}_{k-2} \end{vmatrix} = \begin{vmatrix} \widetilde{W}_2 & \widetilde{W}_1 & \widetilde{W}_0 \\ \widetilde{W}_1 & \widetilde{W}_0 & \widetilde{W}_{-1} \\ \widetilde{W}_0 & \widetilde{W}_{-1} & \widetilde{W}_{-2} \end{vmatrix}.
$$

If we take $n = k+1$, we can get

$$\begin{vmatrix} \widetilde{W}_{k+3} & \widetilde{W}_{k+2} & \widetilde{W}_{k+1} \\ \widetilde{W}_{k+2} & \widetilde{W}_{k+1} & \widetilde{W}_{k} \\ \widetilde{W}_{k+1} & \widetilde{W}_{k} & \widetilde{W}_{k-1} \end{vmatrix} = \begin{vmatrix} 3\widetilde{W}_{k+2} - 3\widetilde{W}_{k+1} + \widetilde{W}_{k} & \widetilde{W}_{k+2} & \widetilde{W}_{k+1} \\ 3\widetilde{W}_{k+1} - 3\widetilde{W}_{k} + \widetilde{W}_{k-1} & \widetilde{W}_{k+1} & \widetilde{W}_{k} \\ 3\widetilde{W}_{k} - 3\widetilde{W}_{k-1} + \widetilde{W}_{k-2} & \widetilde{W}_{k} & \widetilde{W}_{k-1} \end{vmatrix}$$

$$= 3\begin{vmatrix} \widetilde{W}_{k+2} & \widetilde{W}_{k+2} & \widetilde{W}_{k+1} \\ \widetilde{W}_{k+1} & \widetilde{W}_{k+1} & \widetilde{W}_{k} \\ \widetilde{W}_{k} & \widetilde{W}_{k} & \widetilde{W}_{k-1} \end{vmatrix} - 3\begin{vmatrix} \widetilde{W}_{k+1} & \widetilde{W}_{k+2} & \widetilde{W}_{k+1} \\ \widetilde{W}_{k} & \widetilde{W}_{k+1} & \widetilde{W}_{k} \\ \widetilde{W}_{k-1} & \widetilde{W}_{k} & \widetilde{W}_{k-1} \end{vmatrix}$$

$$+ \begin{vmatrix} \widetilde{W}_{k} & \widetilde{W}_{k+2} & \widetilde{W}_{k+1} \\ \widetilde{W}_{k-1} & \widetilde{W}_{k+1} & \widetilde{W}_{k} \\ \widetilde{W}_{k-2} & \widetilde{W}_{k} & \widetilde{W}_{k-1} \end{vmatrix}$$

$$= \begin{vmatrix} \widetilde{W}_{k+2} & \widetilde{W}_{k+1} & \widetilde{W}_{k} \\ \widetilde{W}_{k+1} & \widetilde{W}_{k} & \widetilde{W}_{k-1} \\ \widetilde{W}_{k} & \widetilde{W}_{k-1} & \widetilde{W}_{k-2} \end{vmatrix}.$$

Attention that if we take $n < 0$ the proof can be conducted in a similarly. Thus, the proof is concluded. □

From Theorem (4.1), we get following corollary.

COROLLARY 5.

(a): $$\begin{vmatrix} \widetilde{T}_{n+2} & \widetilde{T}_{n+1} & \widetilde{T}_{n} \\ \widetilde{T}_{n+1} & \widetilde{T}_{n} & \widetilde{T}_{n-1} \\ \widetilde{T}_{n} & \widetilde{T}_{n-1} & \widetilde{T}_{n-2} \end{vmatrix} = -(3\varepsilon + 1)$$

(b): $$\begin{vmatrix} \widetilde{T}_{n+2} & \widetilde{T}_{n+1} & \widetilde{T}_{n} \\ \widetilde{T}_{n+1} & \widetilde{T}_{n} & \widetilde{T}_{n-1} \\ \widetilde{T}_{n} & \widetilde{T}_{n-1} & \widetilde{T}_{n-2} \end{vmatrix} = 0.$$

(c): $$\begin{vmatrix} \widetilde{O}_{n+2} & \widetilde{O}_{n+1} & \widetilde{O}_{n} \\ \widetilde{O}_{n+1} & \widetilde{O}_{n} & \widetilde{O}_{n-1} \\ \widetilde{O}_{n} & \widetilde{O}_{n-1} & \widetilde{O}_{n-2} \end{vmatrix} = -8(3\varepsilon + 1).$$

(d): $$\begin{vmatrix} \widetilde{p}_{n+2} & \widetilde{p}_{n+1} & \widetilde{p}_{n} \\ \widetilde{p}_{n+1} & \widetilde{p}_{n} & \widetilde{p}_{n-1} \\ \widetilde{p}_{n} & \widetilde{p}_{n-1} & \widetilde{p}_{n-2} \end{vmatrix} = -27(3\varepsilon + 1).$$

In the following theorem, we define Catalan's identity of dual generalized Guglielmo numbers.

THEOREM 6. *(Catalan's identity) The following identity is true considering all integers n and m*

$$\widetilde{W}_{n+m}\widetilde{W}_{n-m} - \widetilde{W}_{n}^{2} = m^2(A_3^2\left(2\widetilde{\beta} + \widetilde{a}^2m^2 - 2\widetilde{a}^2n^2 - 4n\widetilde{\beta}\right) - 2A_2A_3\left(\widetilde{\beta} + \widetilde{a}^2n\right) - \widetilde{a}^2\left(A_2^2 - 2A_1A_3\right)). \tag{4.2}$$

Proof. the proof can be done easily using identity (3.1). □

Next we give Catalan's identity of dual triangular, Lucas-triangular, Oblong, pentegonal numbers by using above theorem.

We present Catalan's identity of dual triangular numbers.

COROLLARY 7. *(Catalan's identity for the dual triangular numbers) The following identity is true considering all integers n and m*

$$\widetilde{T}_{n+m}\widetilde{T}_{n-m} - \widetilde{T}_n^2 = -m^2(-\frac{1}{4}\widetilde{a}^2\left(-2n + m^2 - 2n^2 - 1\right) + \widetilde{\beta}n).$$

Proof. If we get $\widetilde{W}_n = \widetilde{T}_n$ in Theorem 6)we obtain the result required. □

We give Catalan's identity of dual triangular-Lucas numbers.

COROLLARY 8. *(Catalan's identity for the dual Lucas-triangular numbers) For all integers n and m, the following identity holds*

$$\widetilde{H}_{n+m}\widetilde{H}_{n-m} - \widetilde{H}_n^2 = 0.$$

Proof. If we get $\widetilde{W}_n = \widetilde{H}_n$ in Theorem 6 we obtain the result required. □

We give Catalan's identity of dual oblong numbers.

COROLLARY 9. *(Catalan's identity for the dual oblong numbers) The following identity is true considering all integers n and m*

$$\widetilde{O}_{n+m}\widetilde{O}_{n-m} - \widetilde{O}_n^2 = -m^2\left(-\widetilde{a}^2(-2n + m^2 - 2n^2 - 1) + 4\widetilde{\beta}n\right).$$

Proof. If we get $\widetilde{W}_n = \widetilde{O}_n$ in Theorem 6 we obtain the result required. □

We give Catalan's identity of dual pentegonal numbers.

COROLLARY 10. *(Catalan's identity for the dual pentegonal numbers) The following identity is true considering all integers n and m*

$$\widetilde{p}_{n+m}\widetilde{p}_{n-m} - \widetilde{p}_n^2 = \frac{1}{4}m^2(\widetilde{a}^2\left(6n + 9m^2 - 18n^2 - 1\right) - 12\widetilde{\beta}\,(3n - 2)).$$

Proof. If we get $\widetilde{W}_n = \widetilde{p}_n$ in Theorem 6 we obtain the result required. □

By setting $m = 1$ in Catalan's identity, we obtain Cassini's identity for the dual generalized Guglielmo numbers. Thus, we present the following corollary.

COROLLARY 11. *(Cassini's identity for the dual generalized Guglielmo numbers) For all integers n, the following identities holds.*

- **(a):** $\widetilde{T}_{n+1}\widetilde{T}_{n-1} - \widetilde{T}_n^2 = \frac{1}{4}\widetilde{a}^2\left(-2n - 2n^2\right) - \widetilde{\beta}n.$
- **(b):** $\widetilde{H}_{n+1}\widetilde{H}_{n-1} - \widetilde{H}_n^2 = 0.$
- **(c):** $\widetilde{O}_{n+1}\widetilde{O}_{n-1} - \widetilde{O}_n^2 = \widetilde{a}^2(-2n - 2n^2) - 4\widetilde{\beta}n.$

(d): $\widetilde{p}_{n+1}\widetilde{p}_{n-1}-\widetilde{p}_{n}^{2}=\frac{1}{4}\widetilde{a}^{2}6n-18n^{2}+8-3\widetilde{\beta}\,(3n-2)).$

THEOREM 12. *We assume that n and m are integers, T_n is triangular numbers, the following identity is true:*

$$\widetilde{W}_{m+n}=T_{m-1}\widetilde{W}_{n+2}+(T_{m-3}-3T_{m-2})\widetilde{W}_{n+1}+T_{m-2}\widetilde{W}_{n}. \tag{4.3}$$

Proof. The identity (12) can be proved by mathematical induction on m. First we take $n,m\geqslant 0$. If $m=0$ we get

$$\widetilde{W}_{n}=T_{-1}\widetilde{W}_{n+2}+(T_{-3}-3T_{-2})\widetilde{W}_{n+1}+T_{-2}\widetilde{W}_{n}$$

which is true by seeing that $T_{-1}=0,T_{-2}=1,T_{-3}=3$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
\widetilde{W}_{(k+1)+n} &= 3\widetilde{W}_{n+k}-3\widetilde{W}_{n+k-1}+\widetilde{W}_{n+k-2}\\
&= 3(T_{k-1}\widetilde{W}_{n+2}+(T_{k-3}-3T_{k-2})\widetilde{W}_{n+1}+T_{k-2}\widetilde{W}_{n})\\
&\quad -3(T_{k-2}\widetilde{W}_{n+2}+(T_{k-4}-3T_{k-3})\widetilde{W}_{n+1}+T_{k-3}\widetilde{W}_{n})\\
&\quad +(T_{k-3}\widetilde{W}_{n+2}+(T_{k-5}-3T_{k-4})\widetilde{W}_{n+1}+T_{k-4}\widetilde{W}_{n})\\
&= (3T_{k-1}-3T_{k-2}+T_{k-3})\widetilde{W}_{n+2}+((3T_{k-3}-3T_{k-4}+T_{k-5})\\
&\quad -3(3T_{k-2}-3T_{k-3}+T_{k-4}))\widetilde{W}_{n+1}+(3T_{k-2}-3T_{k-3}+T_{k-4})\widetilde{W}_{n}\\
&= T_{k}\widetilde{W}_{n+2}+(T_{k-2}-3T_{k-1})\widetilde{W}_{n+1}+T_{k-1}\widetilde{W}_{n}\\
&= T_{(k+1)-1}\widetilde{W}_{n+2}+(T_{(k+1)-3}-3T_{(k+1)-2})\widetilde{W}_{n+1}+T_{(k+1)-2}\widetilde{W}_{n}.
\end{aligned}$$

The other cases on n,m the proof can be done easily. Consequently, by mathematical induction on m, this proves (12). □

5. Linear Sum Formulas of Dual Generalized Guglielmo Numbers

In this section we give some details summation formulas for dual hyperbolic generalized Guglielmo numbers, covering cases with positive and negative subscripts.

PROPOSITION 13. *For the generalized Guglielmo numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n}W_{k}=\frac{1}{12}\,(n+1)\left(\left(2n^{2}-2n\right)W_{2}-2\left(2n^{2}-5n\right)W_{1}+\left(2n^{2}-8n+12\right)W_{0}\right).$

(b): $\sum_{k=0}^{n}W_{k+1}=\frac{1}{12}\,(n+1)\left(\left(2n^{2}+4n\right)W_{2}-2\left(2n^{2}+n-6\right)W_{1}+\left(2n^{2}-2n\right)W_{0}\right).$

Proof. For the proof, see Soykan [20]. □

PROPOSITION 14. *For the generalized Guglielmo numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n}W_{2k}=\frac{1}{12}\,(n+1)\left(\left(8n^{2}-2n\right)W_{2}-2\left(8n^{2}-8n\right)W_{1}+\left(8n^{2}-14n+12\right)W_{0}\right).$

(b): $\sum_{k=0}^{n}W_{2k+1}=\frac{1}{12}\,(n+1)\left(W_{2}\left(8n^{2}+10n\right)-2W_{1}\left(8n^{2}+4n-6\right)+W_{0}\left(8n^{2}-2n\right)\right).$

(c): $\sum_{k=0}^{n} W_{2k+2} = \frac{1}{12}(n+1)\left(\left(8n^2+22n+12\right)W_2 - 2\left(8n^2+16n\right)W_1 + \left(8n^2+10n\right)W_0\right)$.

Proof. For the proof, see Soykan [20]. □

PROPOSITION 15. *For the generalized Guglielmo numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_{-k} = \frac{1}{12}(n+1)\left(\left(2n^2+4n\right)W_2 - 2\left(2n^2+7n\right)W_1 + \left(2n^2+10n+12\right)W_0\right)$.

(b): $\sum_{k=0}^{n} W_{-k+1} = \frac{1}{12}(n+1)\left(\left(2n^2-2n\right)W_2 - 2\left(2n^2+n-6\right)W_1 + \left(2n^2+4n\right)W_0\right)$.

Proof. For the proof, see Soykan [20]. □

PROPOSITION 16. *For the generalized Guglielmo numbers, we have the following formulas:*

(a): $\sum_{k=0}^{n} W_{-2k} = \frac{1}{12}(n+1)\left(\left(8n^2+10n\right)W_2 - 2\left(8n^2+16n\right)W_1 + \left(8n^2+22n+12\right)W_0\right)$.

(b): $\sum_{k=0}^{n} W_{-2k+1} = \frac{1}{12}(n+1)\left(\left(8n^2-2n\right)W_2 - 2\left(8n^2+4n-6\right)W_1 + \left(8n^2+10n\right)W_0\right)$.

(c): $\sum_{k=0}^{n} W_{-2k+2} = \frac{1}{12}(n+1)\left(\left(8n^2-14n+12\right)W_2 - 2\left(8n^2-8n\right)W_1 + \left(8n^2-2n\right)W_0\right)$.

Proof. For the proof, see Soykan [20]. □

Now, we will introduce the formulas that allow us to find the sum of dual generalized Guglielmo numbers.

THEOREM 17. *For $n \geq 0$, dual generalized Guglielmo numbers have the following formulas:*

(a): $\sum_{k=0}^{n} \widetilde{W}_k = \frac{1}{6}(n+1)((-n+\varepsilon n^2+2\varepsilon n+n^2)W_2 + (6\varepsilon+5n-2\varepsilon n^2-\varepsilon n-2n^2)W_1 + (-4n+\varepsilon n^2-\varepsilon n+n^2+6)W_0)$.

(b): $\sum_{k=0}^{n} \widetilde{W}_{2k} = \frac{1}{6}(n+1)((-n+4\varepsilon n^2+5\varepsilon n+4n^2)W_2 + (6\varepsilon+8n-8\varepsilon n^2-4\varepsilon n-8n^2)W_1 + (-7n+4\varepsilon n^2-\varepsilon n+4n^2+6)W_0)$.

(c): $\sum_{k=0}^{n} \widetilde{W}_{2k+1} = \frac{1}{6}(n+1)((6\varepsilon+5n+4\varepsilon n^2+11\varepsilon n+4n^2)W_2 + (6-8\varepsilon n^2-16\varepsilon n-8n^2-4n)W_1 + (-n+4\varepsilon n^2+5\varepsilon n+4n^2)W_0)$.

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^{n} \widetilde{W}_k = \sum_{k=0}^{n} W_k + \varepsilon \sum_{k=0}^{n} W_{k+1}$$

and using Proposition (13) the proof can be done easily.

(b): Note that using (2.1), we get

$$\sum_{k=0}^{n} \widetilde{W}_{2k} = \sum_{k=0}^{n} W_{2k} + \varepsilon \sum_{k=0}^{n} W_{2k+1}$$

and using Proposition (14) the proof can be done easily.

(c): Note that using (2.1), we get

$$\sum_{k=0}^{n} \widetilde{W}_{2k+1} = \sum_{k=0}^{n} W_{2k+1} + \varepsilon \sum_{k=0}^{n} W_{2k+2}$$

and using Proposition (14) the proof can be done easily. □

As a special case of the theorem 17 (a), we present following corollary.

COROLLARY 18.

(a): $\sum_{k=0}^{n}\widetilde{T}_k = \frac{1}{6}(n+1)(6\varepsilon+(5\varepsilon+2)n+(\varepsilon+1)n^2)$.

(b): $\sum_{k=0}^{n}\widetilde{H}_k = (3\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widetilde{O}_k = \frac{1}{6}(n+1)(12\varepsilon+(10\varepsilon+4)n+(2\varepsilon+2)n^2)$.

(d): $\sum_{k=0}^{n}\widetilde{p}_k = \frac{1}{6}(n+1)(6\varepsilon+9\varepsilon n+(3\varepsilon+3)n^2)$.

As a special case of the theorem 17 (b), we present following corollary.

COROLLARY 19.

(a): $\sum_{k=0}^{n}\widetilde{T}_{2k} = \frac{1}{6}(n+1)(6\varepsilon+(5+11\varepsilon)n+(4+4\varepsilon)n^2)$.

(b): $\sum_{k=0}^{n}\widetilde{H}_{2k} = (3\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widetilde{O}_{2k} = \frac{1}{6}(n+1)(12\varepsilon+(10+22\varepsilon)n+(8+8\varepsilon)n^2)$.

(d): $\sum_{k=0}^{n}\widetilde{p}_{2k} = \frac{1}{6}(n+1)(6\varepsilon+(3+21\varepsilon)n+(12+12\varepsilon)n^2)$.

As a special case of the theorem 17 (c), we present following corollary.

COROLLARY 20.

(a): $\sum_{k=0}^{n}\widetilde{T}_{2k+1} = \frac{1}{6}(n+1)((6+18\varepsilon)+(11+17\varepsilon)n+(4+4\varepsilon)n^2)$.

(b): $\sum_{k=0}^{n}\widetilde{H}_{2k+1} = (3\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widetilde{O}_{2k+1} = \frac{1}{6}(n+1)((12+36\varepsilon)+(22+34\varepsilon)n+(8+8\varepsilon)n^2)$.

(d): $\sum_{k=0}^{n}\widetilde{p}_{2k+1} = \frac{1}{6}(n+1)((6+30\varepsilon)+(21+39\varepsilon)n+(12+12\varepsilon)n^2)$.

Now, we present the formula that yield the summation formulas of the generalized Guglielmo numbers with negative subscripts.

THEOREM 21. *For $n \geq 0$, dual generalized Guglielmo numbers have the following formulas:*

(a): $\sum_{k=0}^{n}\widetilde{W}_{-k} = \frac{1}{6}(n+1)((2n+\varepsilon n^2-\varepsilon n+n^2)W_2+(6\varepsilon-7n-2\varepsilon n^2-\varepsilon n-2n^2)W_1+(5n+\varepsilon n^2+2\varepsilon n+n^2+6)W_0)$.

(b): $\sum_{k=0}^{n}\widetilde{W}_{-2k} = \frac{1}{6}(n+1)((5n+4\varepsilon n^2-\varepsilon n+4n^2)W_2+(6\varepsilon-16n-8\varepsilon n^2-4\varepsilon n-8n^2)W_1+(11n+4\varepsilon n^2+5\varepsilon n+4n^2+6)W_0)$.

(c): $\sum_{k=0}^{n}\widetilde{W}_{-2k+1} = \frac{1}{6}(n+1)((6\varepsilon-n+4\varepsilon n^2-7\varepsilon n+4n^2)W_2+(-4n-8\varepsilon n^2+8\varepsilon n-8n^2+6)W_1+(5n+4\varepsilon n^2-\varepsilon n+4n^2)W_0)$.

Proof.

(a): Note that using (2.1), we get

$$\sum_{k=0}^{n}\widetilde{W}_{-k} = \sum_{k=0}^{n}W_{-k}+\varepsilon\sum_{k=0}^{n}W_{-k+1}$$

and using Proposition (15) the proof can be done easily.

(b): Note that using (2.1), we get

$$\sum_{k=0}^{n}\widetilde{W}_{-2k}=\sum_{k=0}^{n}W_{-2k}+\varepsilon\sum_{k=0}^{n}W_{-2k+1}$$

and using Proposition (16) the proof can be done easily.

(c): Note that using (2.1), we get using Proposition (16), we get

$$\sum_{k=0}^{n}\widetilde{W}_{-2k+1}=\sum_{k=0}^{n}W_{-2k+1}+\varepsilon\sum_{k=0}^{n}W_{-2k+2}$$

and using Proposition (16) the proof can be done easily. □

As a special case of the theorem 21 (a), we obtain the following corollary.

Corollary 22.

(a): $\sum_{k=0}^{n}\widetilde{T}_{-k}=\frac{1}{6}(n+1)(6\varepsilon+(-1-4\varepsilon)n+(1+\varepsilon)n^2)$.

(b): $\sum_{k=0}^{n}\widetilde{H}_{-k}=(3\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widetilde{O}_{-k}=\frac{1}{6}(n+1)(12\varepsilon+(-2-8\varepsilon)n+(2+2\varepsilon)n^2)$.

(d): $\sum_{k=0}^{n}\widetilde{p}_{-k}=\frac{1}{2}(n+1)(2\varepsilon+(1-2\varepsilon)n+(1+\varepsilon)n^2)$.

As a special case of the theorem 21 (b), we obtain the following corollary.

Corollary 23.

(a): $\sum_{k=0}^{n}\widetilde{T}_{-2k}=\frac{1}{6}(n+1)(6\varepsilon+(-1-7\varepsilon)n+(4+4\varepsilon)n^2)$.

(b): $\sum_{k=0}^{n}\widetilde{H}_{-2k}=(3\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widetilde{O}_{-2k}=\frac{1}{3}(n+1)(6\varepsilon+(-1-7\varepsilon)n+(4+4\varepsilon)n^2)$.

(d): $\sum_{k=0}^{n}\widetilde{p}_{-2k}=\frac{1}{6}(n+1)((6\varepsilon)+(9-9\varepsilon)n+(12+12\varepsilon)n^2)$.

As a special case of the theorem 21 (c), we obtain the following corollary.

Corollary 24.

(a): $\sum_{k=0}^{n}\widetilde{T}_{-2k+1}=\frac{1}{6}(n+1)((6+18\varepsilon)+(-7-13\varepsilon)n+(4+4\varepsilon)n^2)$.

(b): $\sum_{k=0}^{n}\widetilde{H}_{-2k+1}=(3\varepsilon+3)(n+1)$.

(c): $\sum_{k=0}^{n}\widetilde{O}_{-2k+1}=\frac{1}{3}(n+1)((6+18\varepsilon)+(-7-13\varepsilon)n+(4+4\varepsilon)n^2)$.

(d): $\sum_{k=0}^{n}\widetilde{p}_{-2k+1}=\frac{1}{6}(n+1)((6+30\varepsilon)+(-9-27\varepsilon)n+(12+12\varepsilon)n^2)$.

We will now provide a different theorem that allows us to calculate the finite sum of dual generalized Gaussian numbers.

Theorem 25. *Suppose that x,y,m be integers. The sum formula given below is true*

$$\sum_{k=0}^{m}\widetilde{W}_{xk+y}=(\widetilde{\alpha}A_1+\widetilde{\beta}(A_2+A_3))(m+1)+(\widetilde{\alpha}A_2+2\widetilde{\beta}A_3)\frac{(m+1)}{2}(xm+2y)+\widetilde{a}A_3\frac{(m+1)}{2}(x^2\frac{m(2m+1)}{3}+2xym+2y^2).$$

Proof. For the proof we use Binet's formula of dual generalized Guglielmo numbers and we can write following identity

$$\begin{aligned}\sum_{k=0}^{m}\widetilde{W}_{xk+y} &= \sum_{k=0}^{m}(\widetilde{\alpha}A_1+\widetilde{\beta}(A_2+A_3))+(\widetilde{a}A_2+2\widetilde{\beta}A_3)\sum_{k=0}^{m}(xk+y)+\widetilde{a}A_3\sum_{k=0}^{m}(xk+y)^2\\ &= (\widetilde{\alpha}A_1+\widetilde{\beta}(A_2+A_3))(m+1)+(\widetilde{a}A_2+2\widetilde{\beta}A_3)\frac{(m+1)}{2}(xm+2y)\\ &\quad +\widetilde{a}A_3\frac{(m+1)}{2}(x^2\frac{m(2m+1)}{3}+2xym+2y^2).\end{aligned}$$

Thus, the proof has been completed. □

From the theorem (25) we can write the following corollary.

Corollary 26.

(a): $\sum_{k=0}^{m}\widetilde{T}_{xk+y}=\widetilde{\beta}(m+1)+(\frac{1}{2}\widetilde{\alpha}+\widetilde{\beta})\frac{(m+1)}{2}(xm+2y)+\widetilde{a}\frac{(m+1)}{4}(x^2\frac{m(2m+1)}{3}+2xym+2y^2)$.

(b): $\sum_{k=0}^{m}\widetilde{H}_{xk+y}=3\widetilde{\alpha}(m+1)$.

(c): $\sum_{k=0}^{m}\widetilde{O}_{xk+y}=2\widetilde{\beta}(m+1)+(\widetilde{\alpha}+2\widetilde{\beta})\frac{(m+1)}{2}(xm+2y)+\widetilde{a}\frac{(m+1)}{2}(x^2\frac{m(2m+1)}{3}+2xym+2y^2)$.

(d): $\sum_{k=0}^{m}\widetilde{p}_{xk+y}=\widetilde{\beta}(m+1)+(-\frac{1}{2}\widetilde{\alpha}+3\widetilde{\beta})\frac{(m+1)}{2}(xm+2y)+3\widetilde{a}\frac{(m+1)}{4}(x^2\frac{m(2m+1)}{3}+2xym+2y^2)$.

6. Matrices related with Dual Generalized Guglielmo Numbers

In this section, we give some identities related to matrices using dual generalized Guglielmo Numbers. Here, we examine the triangular sequence $\{T_n\}$ defined by the third-order recurrence relation as follows

$$T_n=3T_{n-1}-3T_{n-2}+T_{n-3}$$

with the initial conditions

$$T_0=0,\ T_1=1,\ T_2=3.$$

We write the third order square matrix A as

$$A=\begin{pmatrix}3&-3&1\\1&0&0\\0&1&0\end{pmatrix}$$

such that $\det A=1$. Then, we have the following Lemma.

Lemma 27. *The following equality holds, for all integers n:*

$$\begin{pmatrix}\widetilde{W}_{n+2}\\\widetilde{W}_{n+1}\\\widetilde{W}_{n}\end{pmatrix}=\begin{pmatrix}3&-3&1\\1&0&0\\0&1&0\end{pmatrix}^n\begin{pmatrix}\widetilde{W}_{2}\\\widetilde{W}_{1}\\\widetilde{W}_{0}\end{pmatrix}. \tag{6.1}$$

Proof. First, we get $n \geq 0$. Lemma (27) can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} \widetilde{W}_2 \\ \widetilde{W}_1 \\ \widetilde{W}_0 \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \widetilde{W}_2 \\ \widetilde{W}_1 \\ \widetilde{W}_0 \end{pmatrix}$$

which is true. We claim that the identity (6.1) given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \widetilde{W}_{k+2} \\ \widetilde{W}_{k+1} \\ \widetilde{W}_k \end{pmatrix} = \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widetilde{W}_2 \\ \widetilde{W}_1 \\ \widetilde{W}_0 \end{pmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \widetilde{W}_2 \\ \widetilde{W}_1 \\ \widetilde{W}_0 \end{pmatrix} &= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \widetilde{W}_2 \\ \widetilde{W}_1 \\ \widetilde{W}_0 \end{pmatrix} \\ &= \begin{pmatrix} 3 & -3 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \widetilde{W}_{k+2} \\ \widetilde{W}_{k+1} \\ \widetilde{W}_k \end{pmatrix} \\ &= \begin{pmatrix} 3\widetilde{W}_{k+2} - 3\widetilde{W}_{k+1} + \widetilde{W}_k \\ \widetilde{W}_{k+2} \\ \widetilde{W}_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} \widetilde{W}_{k+3} \\ \widetilde{W}_{k+2} \\ \widetilde{W}_{k+1} \end{pmatrix}. \end{aligned}$$

For the case $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed.

Note that

$$A^n = \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix}.$$

For the proof and more detail see [23].

THEOREM 28. *If we define the matrices $N_{\widetilde{W}}$ and $E_{\widetilde{W}}$ as follow*

$$N_{\widetilde{W}} = \begin{pmatrix} \widetilde{W}_2 & \widetilde{W}_1 & \widetilde{W}_0 \\ \widetilde{W}_1 & \widetilde{W}_0 & \widetilde{W}_{-1} \\ \widetilde{W}_0 & \widetilde{W}_{-1} & \widetilde{W}_{-2} \end{pmatrix},$$

$$E_{\widetilde{W}} = \begin{pmatrix} \widetilde{W}_{n+2} & \widetilde{W}_{n+1} & \widetilde{W}_n \\ \widetilde{W}_{n+1} & \widetilde{W}_n & \widetilde{W}_{n-1} \\ \widetilde{W}_n & \widetilde{W}_{n-1} & \widetilde{W}_{n-2} \end{pmatrix}.$$

then the following identity is true:

$$A^n N_{\widetilde{W}} = E_{\widetilde{W}}.$$

Proof. For the proof, we can use the following identities

$$A^n N_{\widetilde{W}} = \begin{pmatrix} T_{n+1} & -3T_n + T_{n-1} & T_n \\ T_n & -3T_{n-1} + T_{n-2} & T_{n-1} \\ T_{n-1} & -3T_{n-2} + T_{n-3} & T_{n-2} \end{pmatrix} \begin{pmatrix} \widetilde{W}_2 & \widetilde{W}_1 & \widetilde{W}_0 \\ \widetilde{W}_1 & \widetilde{W}_0 & \widetilde{W}_{-1} \\ \widetilde{W}_0 & \widetilde{W}_{-1} & \widetilde{W}_{-2} \end{pmatrix},$$

$$= \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}$$

where

$$\begin{aligned}
a_{11} &= G\widetilde{W}_2 T_{n+1} + \widetilde{W}_1 (T_{n-1} - 3T_n) + \widetilde{W}_0 T_n, \\
a_{12} &= \widetilde{W}_1 T_{n+1} + \widetilde{W}_0 (T_{n-1} - 3T_n) + \widetilde{W}_{-1} T_n, \\
a_{13} &= \widetilde{W}_0 T_{n+1} + \widetilde{W}_{-1} (T_{n-1} - 3T_n) + \widetilde{W}_{-2} T_n, \\
a_{21} &= \widetilde{W}_2 T_n + \widetilde{W}_1 (T_{n-2} - 3T_{n-1}) + \widetilde{W}_0 T_{n-1}, \\
a_{22} &= \widetilde{W}_1 T_n + \widetilde{W}_0 (T_{n-2} - 3T_{n-1}) + \widetilde{W}_{-1} T_{n-1}, \\
a_{23} &= \widetilde{W}_0 T_n + \widetilde{W}_{-1} (T_{n-2} - 3T_{n-1}) + \widetilde{W}_{-2} T_{n-1}, \\
a_{31} &= \widetilde{W}_2 T_{n-1} + \widetilde{W}_1 (T_{n-3} - 3T_{n-2}) + \widetilde{W}_0 T_{n-2}, \\
a_{32} &= \widetilde{W}_1 T_{n-1} + \widetilde{W}_0 (T_{n-3} - 3T_{n-2}) + \widetilde{W}_{-1} T_{n-2}, \\
a_{33} &= \widetilde{W}_0 T_{n-1} + \widetilde{W}_{-1} (T_{n-3} - 3T_{n-2}) + \widetilde{W}_{-2} T_{n-2}.
\end{aligned}$$

Using the Theorem (12) the proof is done. □

From Theorem (28), the following corollary can be written.

COROLLARY 29.

(a): *We assume that the matrices* $N_{\widetilde{T}}$ *and* $E_{\widetilde{T}}$ *are defined as following*

$$N_T = \begin{pmatrix} \widetilde{T}_2 & \widetilde{T}_1 & \widetilde{T}_0 \\ \widetilde{T}_1 & \widetilde{T}_0 & \widetilde{T}_{-1} \\ \widetilde{T}_0 & \widetilde{T}_{-1} & \widetilde{T}_{-2} \end{pmatrix},$$

$$E_{\widetilde{T}} = \begin{pmatrix} \widetilde{T}_{n+2} & \widetilde{T}_{n+1} & \widetilde{T}_n \\ \widetilde{T}_{n+1} & \widetilde{T}_n & \widetilde{T}_{n-1} \\ \widetilde{T}_n & \widetilde{T}_{n-1} & \widetilde{T}_{n-2} \end{pmatrix},$$

so that the identity given below is true for A^n, $N_{\widetilde{T}}$, $E_{\widetilde{T}}$,

$$A^n N_{\widetilde{T}} = E_{\widetilde{T}},$$

(b): *Let's suppose that the matrices* $N_{\widetilde{H}}$ *and* $E_{\widetilde{H}}$ *are defined as following*

$$N_{\widetilde{H}} = \begin{pmatrix} \widetilde{H}_2 & \widetilde{H}_1 & \widetilde{H}_0 \\ \widetilde{H}_1 & \widetilde{H}_0 & \widetilde{H}_{-1} \\ \widetilde{H}_0 & \widetilde{H}_{-1} & \widetilde{H}_{-2} \end{pmatrix},$$

$$E_{\widetilde{H}} = \begin{pmatrix} \widetilde{H}_{n+2} & \widetilde{H}_{n+1} & \widetilde{H}_n \\ \widetilde{H}_{n+1} & \widetilde{H}_n & \widetilde{H}_{n-1} \\ \widetilde{H}_n & \widetilde{H}_{n-1} & \widetilde{H}_{n-2} \end{pmatrix},$$

so that the identity given below is true for A^n, $N_{\widetilde{H}}$, $E_{\widetilde{H}}$,

$$A^n N_{\widetilde{H}} = E_{\widetilde{O}}.$$

(c): *Let's suppose that the matrices* $N_{\widetilde{O}}$ *and* $E_{\widetilde{O}}$ *are defined as following*

$$N_{\widetilde{O}} = \begin{pmatrix} \widetilde{O}_2 & \widetilde{O}_1 & \widetilde{O}_0 \\ \widetilde{O}_1 & \widetilde{O}_0 & \widetilde{O}_{-1} \\ \widetilde{O}_0 & \widetilde{O}_{-1} & \widetilde{O}_{-2} \end{pmatrix},$$

$$E_{\widetilde{O}} = \begin{pmatrix} \widetilde{O}_{n+2} & \widetilde{O}_{n+1} & \widetilde{O}_n \\ \widetilde{O}_{n+1} & \widetilde{O}_n & \widetilde{O}_{n-1} \\ \widetilde{O}_n & \widetilde{O}_{n-1} & \widetilde{O}_{n-2} \end{pmatrix},$$

so that the identity given below is true for A^n, $N_{\widetilde{O}}$, $E_{\widetilde{O}}$,

$$A^n N_{\widetilde{O}} = E_{\widetilde{O}}.$$

(d): *Let's suppose that the matrices* $N_{\widetilde{p}}$ *and* $E_{\widetilde{p}}$ *are defined as following*

$$N_{\widetilde{p}} = \begin{pmatrix} \widetilde{p}_2 & \widetilde{p}_1 & \widetilde{p}_0 \\ \widetilde{p}_1 & \widetilde{p}_0 & \widetilde{p}_{-1} \\ \widetilde{p}_0 & \widetilde{p}_{-1} & \widetilde{p}_{-2} \end{pmatrix},$$

$$E_{\widetilde{p}} = \begin{pmatrix} \widetilde{p}_{n+2} & \widetilde{p}_{n+1} & \widetilde{p}_{n} \\ \widetilde{p}_{n+1} & \widetilde{p}_{n} & \widetilde{p}_{n-1} \\ \widetilde{p}_{n} & \widetilde{p}_{n-1} & \widetilde{p}_{n-2} \end{pmatrix}.$$

so that the identity given below is true for A^n, $N_{\widetilde{p}}$, $E_{\widetilde{p}}$,

$$A^n N_{\widetilde{p}} = E_{\widetilde{p}}.$$

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