

Original Research Article

Alternative proof of Linear Tangle and Linear Obstacle: An Equivalence Result

Abstract:

Linear-width is a widely recognized and highly valued graph width parameter. The concepts of linear tangle and linear obstacle are dual concepts of linear-width. In this concise paper, we present an alternative proof of the equivalence between linear tangle and linear obstacle.

Keyword: Linear width, Linear tangle, Linear obstacle

1. Introduction

The examination of width parameters holds significant importance in the fields of graph theory and combinatorics, as evidenced by the numerous publications on this topic (e.g., [4, 5, 6, 10, 11, 12, 13, 14, 15, 18, 19, 20, 21]). Linear width, a parameter of graph theory, has been studied in several papers [2, 3, 21].

Linear tangle, a concept initially introduced in reference [1], plays a critical role in determining whether a linear width is at most k (k is a natural number), where $k+1$ represents the order of the tangle (also see references [2, 3, 6]).

Graph searching games are well-known in the fields of graph theory and game theory. A strategy in which the player always emerges victorious is termed a "winning strategy".

This winning strategy is characterized by various width parameters and their dual concepts commonly employed in graph theory (e.g., [6, 7, 8, 9]). For instance, the concept of (k, m) -obstacle on connectivity systems is proposed in reference [6].

In this succinct paper, we present a direct proof for the equivalence between linear tangles and $(k, 1)$ -obstacle, which are referred to as linear obstacle in this paper.

2. Definitions and Notations in this paper

In this section, we provide mathematical definitions for each concept.

2.1 Symmetric Submodular Function

The definition of a symmetric submodular function is given below.

Definition 1: Let X be a finite set. A function $f: X \rightarrow \mathbb{R}$ is called symmetric submodular if it satisfies the following conditions:

- $\forall A \subseteq X, f(A) = f(X/A).$
- $\forall A, B \subseteq X, f(A) + f(B) \geq f(A \cap B) + f(A \cup B).$

In this short paper, a pair (X, f) of a finite set X and a symmetric submodular function f is called a connectivity system. In this paper, we use the notation f for a symmetric submodular function, a finite set X , and a natural number k, m . A set X is k -efficient if $f(X) \leq k$.

2.2. Linear tangle

The definition of a linear tangle is given below.

Definition 2 [1]: A linear tangle of order $k+1$ on a connectivity system (X, f) is a family L of k -efficient subsets of X , satisfying the following axioms:

- (L1) $\emptyset \in L$,
- (L2) For each k -efficient subset A of X , exactly one of A or X/A in L ,
- (L3) If $A, B \in L$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$ holds.

2.3 Linear obstacle: Deep relation to (k, m) -obstacle

The definition of (k, m) -obstacle is shown below.

Definition 3 [10]: In a connectivity system (X, f) , the set family $O \subseteq 2^X$ is called a (k, m) -obstacle if the following axioms hold true:

- (O1) $A \in O, f(A) \leq k$,
- (O2) $A \subseteq B \subseteq X, B \in O, f(A) \leq k \Rightarrow A \in O$,
- (O3) $A, B, C \subseteq X, A \cup B \cup C = X, A \cap B = \emptyset, f(A) \leq k, f(B) \leq k, |C| \leq m \Rightarrow$ either $A \in O$ or $B \in O$.

This paper deals with (k, m) -obstacle for the case where $m = 1$. In this article, we call $(k, 1)$ -obstacle "Linear obstacle" of order $k+1$. Therefore, a linear obstacle is defined as follows:

Definition 4: In a connectivity system (X, f) , the set family $O \subseteq 2^X$ is called a linear obstacle of order $k+1$ if the following axioms hold true:

- (O1) $A \in O, f(A) \leq k$,
- (O2) $A \subseteq B \subseteq X, B \in O, f(A) \leq k \Rightarrow A \in O$,
- (O3) $A, B, C \subseteq X, A \cup B \cup C = X, A \cap B = \emptyset, f(A) \leq k, f(B) \leq k, |C| \leq 1 \Rightarrow$ either $A \in O$ or $B \in O$.

3. Equivalence between linear tangle and linear obstacle

The main result of this paper is below.

Theorem 1. Under the assumption that $f(\{e\}) \leq k$ for every $e \in X$, F is a linear tangle of order $k+1$ on (X, f) iff F is a linear obstacle of order $k+1$ on (X, f) .

Proof of Theorem 1:

Step 1:

Assume F is a linear tangle of order $k+1$ on connectivity system (X, f) . We need to show that F satisfies the axioms of a linear obstacle of order $k+1$.

We show that axiom (O1) holds. From the definition of a linear tangle, F contains k -efficient subsets of X . By the assumption that $f(\{e\}) \leq k$ for every $e \in X$, any set A in F satisfies $f(A) \leq k$, which is consistent with the requirement in (O1).

Next, we show that axiom (O2) holds. Let $A \subseteq B \subseteq X$ such that $B \in F$ and $f(A) \leq k$. We know from the linear tangle definition that exactly one of B or X/B is in F . If $A=B$, then $A \in F$. If $A \neq B$, then $X/A \subseteq X/B$. Since $X/B \notin F$ and $f(X/A) = f(A) \leq k$, $A \in F$. Thus, axiom (O2) holds for F .

Finally, we show that axiom (O3) holds. To demonstrate axiom (O3), we will use proof by contradiction. Assume that axiom (O3) does not hold, which means either $A \notin L$ and $B \notin L$ or $A \in L$ and $B \in L$. We will consider both cases.

Case 1: $A \notin L$ and $B \notin L$

Since $f(B) \leq k$, by axiom (L2), $B \in L$. As $A \cap B = \emptyset$, we have $A \subseteq B$. Since $f(A) \leq k$, by the previously demonstrated axiom (O2), $A \in L$, which leads to a contradiction.

Case 2: $A \in L$ and $B \in L$

In this case, there is a contradiction with axiom (L3).

Thus, in both cases, we arrive at contradictions, which means axiom (O3) must hold.

Step 2:

Assume F is a linear obstacle of order $k+1$ on (X, f) . We need to show that F satisfies the axioms of a linear tangle of order $k+1$.

We show that axiom (L1) holds. Since $f(\emptyset) = f(X) \leq k$, \emptyset is k -efficient, and by axiom (O1), $\emptyset \in F$.

Next, we show that axiom (L2) holds. Let A be a k -efficient subset of X . Since $f(A) \leq k$ and $f(X/A) = f(A) \leq k$, either $A \in F$ or $X/A \in F$ by axiom (O3) with $B=A$, $C=\emptyset$. If both A and X/A were in F , it would contradict (O3) with $B=X/A$ and $C=\emptyset$. Hence, exactly one of A or

X/A is in F .

Finally, we show that axiom (L3) holds. Now, consider the three sets A , B , and C . We have:

1. $A \cup B \cup C = X$,
2. $A \cap B = \emptyset$ (since A and B are both in F and F is a linear obstacle),
3. $f(A) \leq k$, $f(B) \leq k$ (as $A, B \in F$ and F is a linear obstacle),
4. $|C| = 1$.

From the definition of a linear obstacle, either $A \in F$ or $B \in F$ must hold. Since $A, B \in F$ by assumption, this does not contradict the definition. Therefore, $A \cup B \cup \{e\} \neq X$, satisfying axiom (L3).

We have now proven both directions of Theorem 1: F is a linear tangle of order $k+1$ on (X, f) if and only if F is a linear obstacle of order $k+1$ on (X, f) . This proof is completed.

3. Future tasks

In the world of logic, the concept of a weak ideal is known [16, 17]. This is a concept where some of the axioms of an ideal are replaced with weaker ones.

The definition below is based on the concept of an ideal on connectivity system (X, f) as defined in literature [4], and is an extension of the definition of weak ideal using the underlying set and power set, as defined in literature [1]. And an ideal on connectivity system (X, f) is dual concept of branch-width.

Our future goal is to investigate the relationship between weak ideal and various graph parameters and linear tangle. We also plan to study the dual concept of weak ideal, which is the concept of a weak filter.

Definition 5 : In a connectivity system, the set family $W \subseteq 2^X$ is called a weak ideal of order $k+1$ if the following axioms hold true:

- (IB) For every $A \in W$, $f(A) \leq k$.
- (IH) If $A, B \subseteq X$, $f(A) \leq k$, A is a proper subset of B and B belongs to S , then A belongs to S .
- (WIS) If A belongs to S , B belongs to S and $f(A \cup B) \leq k$, then $A \cup B \neq X$.
- (IW) X does not belong to S .

Reference

[1] Daniel Bienstock. Graph searching, path-width, tree-width and related problems (a survey). Reliability of Computer and Communication Networks , Vol.DIMACS. Series in

- Discrete Mathematics and Theoretical Computer Science , pp. 33-50, 1989.
- [2] Takaaki Fujita and Koichi Yamazaki. Linear-width and single ideal. The 20th Anniversary of the Japan Conference on Discrete and Computational Geometry, Graphs, and Games , pp. 110-111, 2017.
- [3] Fujita, T. and Yamazaki, K. (2019) Equivalence between Linear Tangle and Single Ideal. Open Journal of Discrete Mathematics, 9, 7-10. doi: 10.4236/ojdm.2019.91002.
- [4] Yamazaki, Koichi. "Tangle and maximal ideal." WALCOM: Algorithms and Computation: 11th International Conference and Workshops, WALCOM 2017, Hsinchu, Taiwan, March 29–31, 2017, Proceedings 11. Springer International Publishing, 2017.
- [5] Yamazaki, Koichi. "Inapproximability of rank, clique, boolean, and maximum induced matching-widths under small set expansion hypothesis." Algorithms 11.11 (2018): 173.
- [6] Fedor V Fomin and Dimitrios M Thilikos. On the monotonicity of games generated by symmetric submodular functions. Discrete Applied Mathematics, Vol. 131, No. 2, pp. 323–335, 2003.
- [7] Fujita, Takaaki, and Koichi Yamazaki. "Tangle and Ultrafilter: Game Theoretical Interpretation." Graphs and Combinatorics 36.2 (2020): 319-330.
- [8] P. Seymour and R. Thomas. "Graph searching and a min-max theorem for tree-width". Journal of Combinatorial Theory, Series B, Vol. 58, No. 1, pp. 22–23, 1993.
- [9] Isolde Adler. "Games for width parameters and monotonicity". arXiv preprint arXiv:0906.3857, 2009.
- [10] Jim Geelen, Bert Gerards, Neil Robertson, and Geoff Whittle. "Obstructions to branch-decomposition of matroids". Journal of Combinatorial Theory, Series B, Vol. 96, No. 4, pp. 560–570, 2006.
- [11] Koutsonas, Athanassios, Dimitrios M. Thilikos, and Koichi Yamazaki. "Outerplanar obstructions for matroid pathwidth." Discrete Mathematics 315 (2014): 95-101.
- [12] Paul, Christophe, Evangelos Protopapas, and Dimitrios M. Thilikos. "Graph Parameters, Universal Obstructions, and WQO." arXiv preprint arXiv:2304.03688 (2023).
- [13] Reed, Bruce A. "Tree width and tangles: A new connectivity measure and some applications." Surveys in combinatorics (1997): 87-162.
- [14] Diestel, Reinhard. "Ends and tangles." Abhandlungen aus dem Mathematischen Seminar der Universität Hamburg. Vol. 87. No. 2. Berlin/Heidelberg: Springer Berlin Heidelberg, 2017.
- [15] Yamazaki, Koichi, et al. "Isomorphism for graphs of bounded distance width." Algorithmica 24 (1999): 105-127.
- [16] Schlechta, Karl. "Non-Monotonic Logic: Preferential Versus Algebraic Semantics."

David Makinson on Classical Methods for Non-Classical Problems (2014): 167-193.

[17] Dov Gabbay, Karl Schlechta. Conditionals and modularity in general logics. 2010.

[18] Fomin, Fedor V., and Tuukka Korhonen. "Fast fpt-approximation of branchwidth." Proceedings of the 54th Annual ACM SIGACT Symposium on Theory of Computing. 2022.

[19] Kanté, Mamadou Moustapha, et al. "Obstructions for Matroids of Path-Width at most k and Graphs of Linear Rank-Width at most k ." Journal of Combinatorial Theory, Series B 160 (2023): 15-35.

[20] Kobayashi, Yasuaki, and Yu Nakahata. "A Note on Exponential-Time Algorithms for Linearwidth." arXiv preprint arXiv:2010.02388 (2020).

[21] Thilikos, Dimitrios M. "Algorithms and obstructions for linear-width and related search parameters." Discrete Applied Mathematics 105.1-3 (2000): 239-271.