Original Research Article

Linear Tangle and Linear Obstacle: An Equivalence Result

Abstract:

Linear-width is a well-known and highly regarded graph parameter. The concept of linear tangle serves as an obstruction to achieving linear-width. In this concise paper, we present a direct proof for the equivalence between linear tangle and linear obstacle, which are winning strategies used in graph searching games.

Keyword: Linear width, Linear tangle, Linear obstacle

1. Introduction

The examination of width parameters is of importance in the fields of graph theory and combinatorics, as demonstrated by the myriad of publications on this subject. Among these parameters is linear width, which has been examined in various papers (e.g. [4, 5, 6, 10, 11, 12, 13, 14, 15, 18, 19, 20]).

Linear tangle, a concept first introduced in reference [1], plays a crucial role in determining whether a linear width is at most k, where $k+1$ denotes the order of the tangle (see also reference [2,3]).

Graph searching games is a well-known game in the fields of graph theory and game theory. A strategy in which the player always wins is called a "winning strategy". This winning strategy is characterized by various width parameters and their dual concepts commonly used in graph theory (e.g. [6, 7, 8, 9]). For example, the concept of (k,m)-obstacle on connectivity system is proposed in reference [6].

In this concise paper, we present a direct proof for the equivalence between linear tangles and $(k, 1)$-obstacle, which are referred to as linear obstacle in this paper.

2. Definitions

This section provides mathematical definitions of each concept.

2.1 Symmetric Submodular Function

The definition of a symmetric submodular function is given below.

Definition 1 : Let X be a finite set. A function $f: X \to \mathbb{R}$ is called symmetric submodular if it satisfies the following conditions:

· $\forall A \subseteq X$, $f(A) = f(X/A)$.

· $\forall A, B \subseteq X$, $f(A) + f(B) \geq f(A \cap B) + f(A \cup B)$.

In this short paper, a pair (X, f) of a finite set X and a symmetric submodular function f is called a connectivity system. In this paper, we use the notation f for a symmetric submodular function, a finite set X, and a natural number k, m. A set X is k-efficient if $f(X) \leq k$.

2.2. Linear tangle

The definition of a linear tangle is given below.

Definition 2 [1]: A linear tangle of order $k+1$ on a connectivity system (X,f) is a family L of k-efficient subsets of X, satisfying the following axioms:

(L1) $\emptyset \in L$,

(L2) For each k-efficient subset A of X, exactly one of A or X/A in L,

(L3) If $A, B \in L$, $e \in X$, and $f(\{e\}) \leq k$, then $A \cup B \cup \{e\} \neq X$ holds.

2.3 Linear obstacle : Deep relation to *(k,m)*-obstacle

The definition of (k,m)-obstacle is shown below.

Definition 3 [10]: In a connectivity system (X,f), the set family $O \subseteq 2^X$ is called a (k,m)-obstacle if the following axioms hold true:

(O1) $A \in O$, $f(A) \leq k$,

(O2) $A \subseteq B \subseteq X$, $B \in O$, $f(A) \leq k \Rightarrow A \in O$,

(O3) $A, B, C \subseteq X$, $A \cup B \cup C = X$, $A \cap B = \emptyset$, $f(A) \leq k$, $f(B) \leq k$, $|C| \leq m \Rightarrow$ either $A \in O$ or $B \in O$.

This paper deals with (k,m)-obstacle for the case where $m = 1$. In this article, we call $(k,1)$-obstacle "Linear obstacle" of order $k+1$. Therefore, a linear obstacle is defined as follows:

Definition 4: In a connectivity system (X,f), the set family $O \subseteq 2^X$ is called a linear obstacle of order $k + 1$ if the following axioms hold true:

(O1) $A \in O$, $f(A) \leq k$,

(O2) $A \subseteq B \subseteq X$, $B \in O$, $f(A) \leq k \Rightarrow A \in O$, (O3) $A, B, C \subseteq X$, $A \cup B \cup C = X$, $A \cap B = \emptyset$, $f(A) \leq k$, $f(B) \leq k$, $|C| \leq 1 \Rightarrow$ either $A \in O$ or $B \in O$.

3. Equivalence between linear tangle and linear obstacle

The result of this paper is below.

Theorem 1. Under the assumption that $f(\{e\})\leq k$ for every $e\in X$, F is a linear tangle of order $k+1$ on (X,f) iff F is a linear obstacle of order $k+1$ on (X,f) .

Proof of Theorem 1:

Step 1:

Assume F is a linear tangle of order $k+1$ on connectivity system (X,f). We need to show that F satisfies the axioms of a linear obstacle of order $k+1$.

We show that axiom (O1) holds. From the definition of a linear tangle, F contains k-efficient subsets of X. By the assumption that $f(\{e\})\leq k$ for every $e\in X$, any set A in F satisfies $f(A)\leq k$, which is consistent with the requirement in (O1).

Next, we show that axiom (O2) holds. Let $A\subseteq B\subseteq X$ such that $B\in F$ and $f(A)\leq k$. We know from the linear tangle definition that exactly one of B or X/B is in F. If A=B, then $A\in F$. If $A\neq B$, then $X/A\subseteq X/B$. Since $X/B\notin F$ and $f(X/A)=f(A)\leq k$, A∈F. Thus, axiom (O2) holds for F.

Finally, we show that axiom (O3) holds. To demonstrate axiom (O3), we will use proof by contradiction. Assume that axiom (O3) does not hold, which means either $A \notin L$ and $B \notin L$ or $A \in L$ and $B \in L$. We will consider both cases.

Case 1: $A \notin L$ and $B \notin L$

Since $f(B) \leq k$, by axiom (L2), $B \in L$. As $A \cap B = \emptyset$, we have $A \subseteq B$. Since $f(A) \leq k$, by the previously demonstrated axiom (O2), $A \in L$, which leads to a contradiction.

Case 2: $A \in L$ and $B \in L$

In this case, there is a contradiction with axiom (L3).

Thus, in both cases, we arrive at contradictions, which means axiom (O3) must hold.

Step 2:

Assume F is a linear obstacle of order $k+1$ on (X,f). We need to show that F satisfies the axioms of a linear tangle of order $k+1$.

We show that axiom (L1) holds. Since $f(\emptyset) = f(X) \leq k$, $\emptyset$ is k-efficient, and by axiom (O1), $\emptyset\in F$.

Next, we show that axiom (L2) holds. Let A be a k-efficient subset of X. Since $f(A) \leq k$

and $f(X/A) = f(A) \leq k$, either $A \in F$ or $X/A \in F$ by axiom (O3) with $B=A$, $C=\emptyset$. If both A and X/A were in F, it would contradict (O3) with $B=X/A$ and $C=\emptyset$. Hence, exactly one of A or X/A is in F.

Finally, we show that axiom (L3) holds. Now, consider the three sets A, B, and C. We have:

1. $A \cup B \cup C = X$,
2. $A \cap B = \emptyset$ (since A and B are both in F and F is a linear obstacle),
3. $f(A) \leq k$, $f(B) \leq k$ (as $A, B \in F$ and F is a linear obstacle),
4. $|C| = 1$.

From the definition of a linear obstacle, either $A \in F$ or $B \in F$ must hold. Since $A, B \in F$ by assumption, this does not contradict the definition. Therefore, $A \cup B \cup \{e\} \neq X$, satisfying axiom (L3).

We have now proven both directions of Theorem 1: F is a linear tangle of order $k+1$ on (X,f) if and only if F is a linear obstacle of order $k+1$ on (X,f). This proof is completed.

3. Future tasks

In the world of logic, the concept of a weak ideal is known [16, 17]. This is a concept where some of the axioms of an ideal are replaced with weaker ones.

The definition below is based on the concept of an ideal on connectivity system (X, f) as defined in literature [4], and is an extension of the definition of weak ideal using the underlying set and power set, as defined in literature [1].

Our future goal is to investigate the relationship between weak ideal and various graph parameters and linear tangle. We also plan to study the dual concept of weak ideal, which is the concept of a weak filter.

Definition 5 : In a connectivity system, the set family $W \subseteq 2^X$ is called a weak ideal of order $k+1$ if the following axioms hold true:

(IB) For every $A \in W$, $f(A) \leq k$.

(IH) If $A, B \subseteq X$, $f(A) \leq k$, A is a proper subset of B and B belongs to S, then A belongs to S.

(WIS) If A belongs to S, B belongs to S and $f(A \cup B) \leq k$, then $A \cup B \neq X$.

(IW) X does not belong to S.

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