WaterWave Solutions using Lagrangian Function from Hamilton's Principle

Abstract

This paper concerns the development and application of the Lagrangian function which is the difference between kinetic energy and potential energy of the system. Here irrotational, incompressible, inviscid fluid in finite water depth is considered. Our attention is to focus on the problem to solve water wave evolution with Lagrangian function which is obtained from Hamilton's Principle. Then Lagrangian function is expanded under the assumption that the dispersion μ and the nonlinearity ε satisfied $\varepsilon = O(\mu^2)$. Here the Lagrangian function is generalised up to $O(\mu^8)$. The elevation of the free surface should be expanded to μ^4 order to get the Lagrangian function is in μ^8 order. Finally a wave model from Euler- Lagrangian equation of motion has been generalized which follows that the generalized wavevelocity decreases with the large value of time and at very large time, a wave crest and trough will be diminished.

Key words: Water wave equation, Lagrangian function, Laplace equation, Hamilton's principle.

Introduction

"Here we have a good discussion on the principles of different kinds of water wave theory. The governing mathematical equations of Eulerian conservation of mass, momentum, energy are used to describe different forms of water waves. Boussinesq equation represents a shallow water approximation to the exact Laplace problem which incorporates the balance between lowest-order dispersion and lowest-level non-linearity. Many researchers have tried to derive modified forms of the classical Boussinesq equation over last decades and a number of enhanced higher-order Boussinesq equations have been derived improving the dispersion and non-linearity as well as flow

kinematics and dynamics" (e.g., Nwogu(1993), Agnon et al.(1999) ,Madsen et al.(2002,2003) Among these, the formulation of Madsen et al.(2002,2003) is "most capable of treating highly non-linear waves to *kh*=25 for dispersion, with accurate velocity profiles up to kh=12. Dynamics research on Hamilton systems is an important subject in mechanics for a long time". The principles of Hamilton mechanics settled a series of problems effectively that could not be solved by other methods, which showed theoretically the importance of Hamilton mechanics. Whitham (1965) used "fluid dynamics, Hamilton principles and variational principles for water waves and related problems in the theory of nonlinear dispersive waves". Luke (1967) obtained "a Lagrangian function yielding the Laplace's equation and the boundary conditions at the surface and bottom"..Zakharov(1968) showed that "the water elevation and the potential at the free surface are canonical variables when formulating the water-waves problem in Hamiltonian formalism, the Hamilton function being the total energy of the fluid". The mathematical properties of the Hamiltonian formalism for free surface waves was extensively studied by Miles (1977), Milder (1977), Radder (1992) and many other authors. Hou et al.(1998) used "the variational principle to establish a nonlinear equation for shallow water wave evolution". Ambrosi (2000) gave "a Hamilton formulation for surface waves in a layered fluid". Lvov and Tabak (2004) developed "a Hamilton formulation for long internal waves. Hongli et al.(2006) derived water wave solutions using variation method". In this paper solution of water wave equation is derived using Hamilton's principle and then wave model from Euler-Lagrangian equation has been formed.

Methodology

For the seek of solutions of water wave equations different researchers used so many methods that they reached their goal with various models in natural science and play a great role specially in fluid dynamics. Hamilton's Principle for irrotational water waves is used to obtain Lagrange function from which generalized wavevelocity can be derived directly for the given system.

From the theory of an ideal and homogeneous fluid,the vorticity $\nabla \times v$ is a property associated with the fluid elements. It is carried along by the fluid motion. This implies that if a particular fluid element had zero vorticity initially, it will always have zero vorticity. The main property of a wave is its ability to transport information, energy and momentum over considerable distances without transport of matter. Thus the velocity field associated with the wave is irrotational and given by a velocity potential, φ, which according to the above equation satisfies the Laplace equation , $\nabla^2 \varphi = 0$

Now the Hamilton's Principle for irrotational water waves free of side conditions is

$$\delta \int_{t_1}^{t_2} L\,dt = 0, \qquad\qquad (1)$$

with Lagrange function $L = \int_{-H}^{\eta(x,t)} \left\{ \varphi_t + \frac{1}{2}|\nabla\varphi|^2 + gz \right\} dz$ (2)

Here t is time, z is the vertical coordinate, x is the horizontal coordinate, x-axis represents undisturbed surface with

constant depth H, $\varphi(x,z,t)$ is the velocity potential, here $\eta(x,t)$ is the elevation of the free surface, and g is the

acceleration of gravity.

Then, we have variation of φ within the flow region

$$\left. \begin{array}{l} \nabla^2\phi = 0, \qquad -H < z < \eta \\ \eta_t + \varphi_x\eta_x - \varphi_z = 0, \quad z = \eta(x,t) \\ \varphi_z = 0, \quad z = -H. \end{array} \right\} \qquad (3)$$

The variation of η gives the dynamical boundary condition on the free surface:

$$\phi_t + \frac{1}{2}(\nabla\varphi)^2 + gz = 0, \quad z = \eta(x,t). \qquad (4)$$

Formulation

We introduce the following non-dimensional variables:

$$x^* = kx, \ z^* = \frac{z}{H}, \ t^* = kct, \ \eta^* = \frac{\eta}{a}, \ \varphi^* = \frac{kH}{ac}\varphi, \quad L^* = \frac{L}{gH^2}$$

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where k and a are wave number and wave amplitude respectively, and $c = \sqrt{gH}$ is typical wave speed for shallow water.

In terms of these non-dimensional variables, above equation can be rewritten as

$$L = \int_{-1}^{\varepsilon\eta}\left\{\varepsilon\frac{\partial\varphi}{\partial t} + \frac{\varepsilon^2}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{\mu^2}\left(\frac{\partial\varphi}{\partial z}\right)^2\right] + z\right\}dz \qquad (5)$$

$$\mu^2\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} = 0, \quad -1 < z < \varepsilon\eta \qquad (6)$$

$$\eta_t + \varepsilon\varphi_x\eta_x - \frac{1}{\mu^2}\varphi_z = 0, \ z = \varepsilon\eta \qquad (7)$$

$$\frac{\partial\varphi}{\partial z} = 0, \ z = -1 \qquad (8)$$

$$\frac{\partial\varphi}{\partial t} + \frac{\varepsilon}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{\mu^2}\left(\frac{\partial\varphi}{\partial z}\right)^2 + \eta\right] = 0, \quad z = \varepsilon\eta. \qquad (9)$$

For the convenience of our calculations, the asterisks have been omitted and then $\mu = kH$ and $\varepsilon = \dfrac{a}{H}$ stand for the dispersion and nonlinearity, respectively.

Under linear approximation, Eqs. (6) to (9) become

$$\mu^2\frac{\partial^2\varphi}{\partial x^2} + \frac{\partial^2\varphi}{\partial z^2} = 0, \qquad -1 < z < 0 \qquad (10)$$

$$\eta_t - \frac{1}{\mu^2}\varphi_z = 0, \quad z = 0 \qquad (11)$$

$$\frac{\partial\varphi}{\partial z} = 0, \qquad z = -1 \qquad (12)$$

$$\frac{\partial\varphi}{\partial t} + \eta = 0, \qquad z = 0. \qquad (13)$$

Here we use the following coordinate transforms to describe the behaviors of water wave varying slowly with time,

$$\begin{cases} X = x - \sigma t \\ \tau = \mu^2 t \end{cases}$$

where $\sigma^2 = \dfrac{1}{\mu} \tanh \mu$.

In terms of these variables, Eqs. (10) to (13) become

$$\mu^2 \frac{\partial^2 \varphi}{\partial X^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0, \qquad \frac{\partial X}{\partial x} = 1 \qquad (14)$$

$$\frac{\partial \eta}{\partial t} - \frac{1}{\mu^2} \phi_z = 0$$

$$\therefore \sigma \frac{\partial \eta}{\partial X} + \frac{1}{\mu^2} \varphi_z = 0, \qquad z = 0 \qquad (15)$$

$$\frac{\partial \varphi}{\partial z} = 0, \qquad z = -1 \qquad (16)$$

$$\frac{\partial \phi}{\partial X} \frac{\partial X}{\partial t} + \eta = 0$$

$$\therefore -\sigma \frac{\partial \varphi}{\partial X} + \eta = 0, \qquad z = 0. \qquad (17)$$

Solutions

The solution of Eq. (14) in the bottom boundary Eq. (16) is

$$\varphi_j = q_j \cos X \cosh \mu(z+1) \qquad (18)$$

where q_j is a constant.

We have the following expression from the dynamical boundary condition (17)

$$\sigma q_j \sin X \cosh \mu(z+1) + \eta_j = 0$$

$$\therefore \eta_j = -q_j \sigma \cosh \mu(z+1) \sin X$$

$$\therefore \eta_j = -q_j \left(\frac{\tanh \mu}{\mu} \right)^{\frac{1}{2}} \cosh \mu(z+1) \sin X. \qquad (19)$$

Now we consider, $\varphi = \displaystyle\sum_{n=0}^{\infty} \varphi_n (z+1)^n.$ \qquad (20)

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Now using these from Eq. (6) we obtain

$$\sum_{n=0}^{\infty}\left[\mu^2 \frac{\partial^2 \varphi_n}{\partial x^2} + (n+2)(n+1)\varphi_{n+2}\right](z+1)^n = 0 \qquad (21)$$

Since z be an arbitrary value in $(-1, \varepsilon\eta)$, so each coefficient in power of (z + 1) must be zero, thus

$$\varphi_{n+2} = -\frac{\mu^2}{(n+2)(n+1)}\frac{\partial^2 \varphi_n}{\partial x^2}, \quad n = 0,1,2,..... \quad (22)$$

On the other hand, substituting Eq. (20) into Eq. (8) yields $\varphi_1 = 0$. Therefore, for all odds, $\varphi_n = 0$, i.e.,

$$\varphi_1 = \varphi_3 = \varphi_5 = ... = 0.$$

Supposing that $\varphi_0 = \Phi$, we have

$$\varphi_{2n} = (-1)^n \frac{\mu^{2n}}{(2n)!}\frac{\partial^{2n}\Phi}{\partial x^{2n}}, \qquad \varphi_{2n+1} = 0, \ n = 0,1,2,3..............$$

Now, the expression of velocity potential φ is obtained:

$$\varphi = \sum_{n=0}^{\infty}(-1)^n \frac{\partial^{2n}\Phi}{\partial x^{2n}}\frac{\mu^{2n}}{(2n)!}(z+1)^{2n}$$

By linear approximation, we also consider $\quad \Phi = q_j \cos X.$ \hspace{2cm} (23)

Therefore, the velocity potential φ can be found to be: $\varphi(x,t,z) = q_j \cos X \cosh \mu(z+1).$

$$\varphi(x, z, t) = \Phi \cosh \mu(z+1), \qquad by \quad using \ (23)$$

Now

$$\frac{\xi^2}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\frac{1}{\mu^2}\left(\frac{\partial\varphi}{\partial z}\right)^2\right]=-\frac{\xi^2}{4}q_j^{~2}\cos 2X+\frac{\xi^2}{4}q_j^{~2}\cos 2h\mu(z+1).$$

$$L=\int_{-1}^{\varepsilon\eta}\left\{\frac{\partial\varphi}{\partial t}+\frac{\varepsilon^2}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\frac{1}{\mu^2}\left(\frac{\partial\varphi}{\partial z}\right)^2\right]+z\right\}dz$$

$$=\int_{-1}^{\varepsilon\eta}\varepsilon\frac{\partial\phi}{\partial t}\cosh\mu(z+1)dz-\frac{\mu^2}{4}q_j^{~2}\int_{-1}^{\varepsilon\eta}\cos 2Xdz+\frac{\varepsilon^2}{4}q_j^{~2}\int_{-1}^{\varepsilon\eta}\cos 2h\mu(z+1)dz+\int_{-1}^{\varepsilon\eta}zdz$$

$$=\frac{\varepsilon}{\mu}\frac{\partial\phi}{\partial t}\sinh\mu(\varepsilon\eta+1)-\frac{\varepsilon^2}{4}q_j^{~2}\cos 2X(1+\varepsilon\eta)+\frac{\varepsilon^2}{8\mu}q_j^{~2}\sin 2h\mu(1+\varepsilon\eta)+\frac{1}{2}(\varepsilon\eta)^2-\frac{1}{2}.$$

Neglecting the constant term $\dfrac{1}{2}$, the above equation yields

$$L=\frac{\varepsilon}{\mu}\frac{\partial\Phi}{\partial t}\sinh\mu(1+\varepsilon\eta)-\frac{\varepsilon^2}{4}q_j^{~2}\cos 2X(1+\varepsilon\eta)+\frac{\varepsilon^2}{8\mu}q_j^{~2}\sinh 2\mu(1+\varepsilon\eta)+\frac{1}{2}(\varepsilon\eta)^2 \qquad (24)$$

The case of $\varepsilon=O(\mu^2)$, was considered by Benjamin(1967)and Whitham (1967) who obtained the Korteweg de

Vries (KdV) equation. Here we also consider the case, and expand Lagrangian function up to $O(\mu^8)$order

$$L=\varepsilon\left(1-\frac{1}{6}\mu^2+\frac{19}{360}\mu^4-\frac{401}{630\times 24}\mu^6\right)q_j\sin X\left[1+\frac{\mu^2}{6}+\frac{\mu^4}{5!}+\frac{\mu^6}{7!}+\varepsilon\left(1+\frac{\mu^2}{2}+\frac{\mu^4}{24}\right)\eta+\frac{1}{2}\varepsilon^2\mu^2\eta^2\right]$$

$$+\varepsilon\mu^2 q_j'\cos X\left[1+\frac{\mu^2}{6}+\frac{\mu^4}{120}+\varepsilon\left(1+\frac{\mu^2}{2}\right)\eta\right]-\frac{\varepsilon^2}{4}q_j^{~2}\cos 2X(1+\varepsilon\eta)$$

$$+\frac{\varepsilon^2}{4}q_j^{~2}\left[1+\frac{2}{3}\mu^2+\frac{2}{15}\mu^4+\varepsilon\eta+2\varepsilon\mu^2\eta+\frac{2}{3}\varepsilon\mu^4\eta\right]+\frac{1}{2}\varepsilon^2\eta^2+O(\mu^8) \qquad (25)$$

Hou et al.(1998) used the lowest-order of η in their article.

Let $\quad\eta=a_1+a_2\mu^2+a_3\mu^4+a_4\mu^6+O(\mu^6)$

Substituting it into Eq. (25), we can see that we should expand η to μ^4,

$$\eta=a_1+a_2\mu^2+a_3\mu^4+O(\mu^6)$$

Based on the dynamical boundary condition of the free surface Eqn. (9), we have

$$\eta = -\frac{\partial \varphi}{\partial t}\bigg|_{z=\varepsilon\eta} - \frac{\varepsilon}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{\mu^2}\left(\frac{\partial \varphi}{\partial z}\right)^2\right]\bigg|_{z=\varepsilon\eta}$$

$$\therefore a_1 + a_2\mu^2 + a_3\mu^4 + O(\mu^6) = -q_j\sin X - \frac{\mu^2}{3}q_j\sin X - \mu^2\dot{q}_j(\tau)\cos X - \frac{\varepsilon}{2}q_j^2\sin^2 X$$

$$+ \varepsilon\mu^2\left[q_j^2\sin^2 X - \frac{1}{2}q_j^2(\tau)\right] - \frac{\mu^4}{90}q_j\sin X - \frac{\mu^4}{2}\dot{q}_j\cos X + O(\mu^6) \qquad (26)$$

From Eq. (26), equating the coefficients of constant, μ^2 and μ^4 terms, we have

$$a_1 = -q_j\sin X$$

$$a_2 = -\dot{q}_j\cos X - \frac{1}{3}q_j\sin X - \frac{1}{2}q_j^2\sin^2 X \qquad (27)$$

$$a_3 = -\frac{1}{2}\dot{q}_j\cos X - \frac{1}{2}q_j^2 + q_j^2\sin^2 X - \frac{1}{90}q_j\sin X$$

Eq.(1) can be rewritten as $\delta\int L\,d\tau = 0$

$$\text{where } I\left[\dot{q}_j, q_j\right] = \frac{1}{2\pi}\int_0^{2\pi} L\,dX. \qquad (28)$$

From eq. (28) we have the Lagrangian

$$L\left(q_j, \dot{q}_j\right) = -\frac{1}{4}\varepsilon^2\mu^4\dot{q}_j^2 - \frac{3}{64}\varepsilon^4 q_j^4. \qquad (29)$$

Obviously, Lagrangian is a function of generalized coordinates and generalized velocity.

Then from Lagrange's Equation of motion

$$\ddot{q}_j - \frac{3}{8}\frac{\varepsilon^2}{\mu^4}q_j^3 = 0 \qquad (30)$$

Solving this for $\varepsilon = \mu^2$, $\qquad q_j = \dfrac{4}{\sqrt{3}}\tau^{-1}$ $\qquad\qquad$ (31)

From Eq. (31) it is obvious that generalized velocities be $\dot{q}_j = -\dfrac{4}{\sqrt{3}}\tau^{-2}$ $\qquad$ (32)

which gives a wave model from Euler- Lagrangian equation. Thus the generalized velocity decreases with the large value of time. But at larger time generalized velocity will be diminished and then wave crest and trough will be flattened.

Conclusion

Firstly, we have derived the Lagrangian function from Hamilton's principle which is expanded up to $O(\mu^8)$, then after a few simplifications and transformations, water wave equations are solved. From Lagrange's equation of motion, it is seen that the generalized wave velocity decreases with the large value of time and the velocity will be diminished when time is large enough. Again from the above discussion of non-dimensional free surface profiles, wave crest and trough will be flattened at larger time. The results which we obtained in the above topics can be studied for the effect of wavevelocity in deep water andalso we can show that wave crest and trough will be flattened at very very large time.

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