Water Wave Solutions Obtained by Hamilton's Principle

Abstract

This paper concerns the development and application of the Lagrangian function which is the difference between kinetic energy and potential energy of the system. Here irrotational, incompressible, inviscid fluid in finite water depth is considered. . Then Lagrangian function is expanded under the assumption that the dispersion μ and the nonlinearity ε satisfied $\varepsilon = O(\mu^2)$ .Here the Lagrangian function is generalised up to $O(\mu^8)$. The elevation of the free surface should be expanded to μ^4 order to get the Lagrangian function is in μ^8 order. Here our attention is focused on the problem to solve using Hamilton's Principle for water wave evolution and then we show wave crest and trough will be flattened at larger time.

Key words: Water wave equation, Lagrangian function, Laplace equation.

Introduction

Here we have a good discussion on the principles of different kinds of water wave theory. The governing mathematical equations of Eulerian conservation of mass, momentum, energy are used to describe different forms of water waves. Boussinesq equation represents a shallow water approximation to the exact Laplace problem which incorporates the balance between lowest-order dispersion and lowest-level non-linearity. Many researchers have tried to derive modified forms of the classical Boussinesq equation over last decades and a number of enhanced higher-order Boussinesq equations have been derived improving the dispersion and non-linearity as well as flow kinematics and dynamics (e.g., Nwogu(1993), Agnon et al.(1999) ,Madsen et al.(2002,2003) Among these, the formulation of Madsen et al.(2002,2003) is most capable of treating highly non-linear waves to *kh*=25 for dispersion, with accurate velocity profiles up to kh=12. Dynamics research on Hamilton systems is an important subject in mechanics for a long time. The principles of Hamilton mechanics settled a series of problems effectively that could not be solved by other methods, which showed theoretically the importance of Hamilton mechanics. Whitham (1965) used fluid dynamics, Hamilton principles and variational principles for water waves and related problems in the theory of nonlinear dispersive waves. Luke (1967) obtained a Lagrangian function yielding the Laplace's equation and the boundary conditions at the surface and bottom.. Zakharov(1968) showed that the water elevation and the potential at the free surface are canonical variables when formulating the water-waves problem in Hamiltonian formalism, the Hamilton function being the total energy of the fluid. The mathematical properties of the Hamiltonian formalism for free surface waves was extensively studied by Miles (1977), Milder (1977), Radder (1992) and many other authors. Hou et al.(1998) used the variational principle to establish a nonlinear equation for shallow water wave evolution. Ambrosi (2000) gave a Hamilton formulation for surface waves in a layered fluid. Lvov and Tabak (2004) developed a Hamilton formulation for long internal waves. Hongli et al.(2006) derived water wave solutions using variation method. In this paper solution of water wave equation is derived using Hamilton's principle and then wave model from Euler-Lagrangian equation has been formed.

Mathematical equations

From the theory of an ideal and homogeneous fluid ,the vorticity $\nabla \times v$ is a property associated with the fluid elements. It is carried along by the fluid motion. This implies that if a particular fluid element had zero vorticity initially, it will always have zero vorticity. The main property of a wave is its ability to transport information, energy and momentum over considerable distances without transport of matter. Thus the velocity field associated with the wave is irrotational and given by a velocity potential, φ , which according to the above equation satisfies the Laplace equation

$$\nabla^2 \varphi = 0$$

Now the Hamilton's Principle for irrotational water waves free of side conditions is

$$\delta \int_{t_1}^{t_2} L dt = 0, \qquad (1)$$

with Lagrange function $L = \int_{-H}^{\eta(x,t)} \left\{ \varphi_t + \frac{1}{2} |\nabla \varphi|^2 + gz \right\} dz$ (2)

Here t is time, z is the vertical coordinate, x is the horizontal coordinate, x-axis represents undisturbed surface with constant depth H, $\varphi(x,z,t)$ is the velocity potential, here $\eta(x,t)$ is the elevation of the free surface, and g is the acceleration of gravity.

Then, we have variation of φ within the flow region

$$\left.\begin{aligned} &\nabla^2\phi = 0, \qquad -H < z < \eta \\ &\eta_t + \varphi_x\eta_x - \varphi_z = 0, \;\; z = \eta(x,t) \\ &\varphi_z = 0, \;\; z = -H. \end{aligned}\right\} \tag{3}$$

The variation of η gives the dynamical boundary condition on the free surface:

$$\phi_t + \frac{1}{2}(\nabla\varphi)^2 + gz = 0, \quad z = \eta(x,t). \tag{4}$$

Formulation

We introduce the following non-dimensional variables:

$$x^* = kx,\; z^* = \frac{z}{H},\;\; t^* = kct,\;\; \eta^* = \frac{\eta}{a},\;\; \varphi^* = \frac{kH}{ac}\varphi, \qquad L^* = \frac{L}{gH^2}$$

where k and a are wave number and wave amplitude respectively, and $c = \sqrt{gH}$ is typical wave speed for shallow water.

In terms of these non-dimensional variables, above equation can be rewritten as

$$L = \int_{-1}^{\varepsilon\eta} \left\{ \varepsilon \frac{\partial \varphi}{\partial t} + \frac{\varepsilon^2}{2} \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{\mu^2} \left(\frac{\partial \varphi}{\partial z} \right)^2 \right] + z \right\} dz \qquad (5)$$

$$\mu^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0, \quad -1 < z < \varepsilon\eta \qquad (6)$$

$$\eta_t + \varepsilon\varphi_x \eta_x - \frac{1}{\mu^2} \varphi_z = 0, \ z = \varepsilon\eta \qquad (7)$$

$$\frac{\partial \varphi}{\partial z} = 0, \ z = -1 \qquad (8)$$

$$\frac{\partial \varphi}{\partial t} + \frac{\varepsilon}{2} \left[\left(\frac{\partial \varphi}{\partial x} \right)^2 + \frac{1}{\mu^2} \left(\frac{\partial \varphi}{\partial z} \right)^2 + \eta \right] = 0, \quad z = \varepsilon\eta. \qquad (9)$$

For the convenience of our calculations, the asterisks have been omitted and then $\mu = kH$ and $\varepsilon = \frac{a}{H}$ stand for the dispersion and nonlinearity, respectively.

Under linear approximation, Eqs. (6) to (9) become

$$\mu^2 \frac{\partial^2 \varphi}{\partial x^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0, \quad -1 < z < 0 \qquad (10)$$

$$\eta_t - \frac{1}{\mu^2} \varphi_z = 0, \quad z = 0 \qquad (11)$$

$$\frac{\partial \varphi}{\partial z} = 0, \quad z = -1 \qquad (12)$$

$$\frac{\partial \varphi}{\partial t} + \eta = 0, \quad z = 0. \qquad (13)$$

Here we use the following coordinate transforms to describe the behaviors of water wave varying slowly with time,

$$\begin{cases} X = x - \sigma t \\ \tau = \mu^2 t \end{cases}$$

where $\sigma^2 = \frac{1}{\mu} \tanh \mu.$

In terms of these variables, Eqs. (10) to (13) become

$$\mu^2 \frac{\partial^2 \varphi}{\partial X^2} + \frac{\partial^2 \varphi}{\partial z^2} = 0, \qquad \frac{\partial X}{\partial x} = 1 \qquad (14)$$

$$\frac{\partial \eta}{\partial t} - \frac{1}{\mu^2} \phi_z = 0$$

$$\therefore \sigma \frac{\partial \eta}{\partial X} + \frac{1}{\mu^2} \varphi_z = 0, \qquad z = 0 \qquad (15)$$

$$\frac{\partial \varphi}{\partial z} = 0, \qquad z = -1 \qquad (16)$$

$$\frac{\partial \phi}{\partial X} \frac{\partial X}{\partial t} + \eta = 0$$

$$\therefore -\sigma \frac{\partial \varphi}{\partial X} + \eta = 0, \qquad z = 0. \qquad (17)$$

Solutions

The solution of Eq. (14) in the bottom boundary Eq. (16) is

$$\varphi_j = q_j \cos X \cosh \mu (z+1) \qquad (18)$$

where q_j is a constant.

We have the following expression from the dynamical boundary condition (17)

$$\sigma q_j \sin X \cosh \mu (z+1) + \eta_j = 0$$
$$\therefore \eta_j = -q_j \sigma \cosh \mu (z+1) \sin X$$

$$\therefore \eta_j = -q_j \left(\frac{\tanh \mu}{\mu} \right)^{\frac{1}{2}} \cosh \mu (z+1) \sin X. \qquad (19)$$

Now we consider, $\varphi = \sum_{n=0}^{\infty} \varphi_n (z+1)^n.$ (20)

Now using these from Eq. (6) we obtain

$$\sum_{n=0}^{\infty}\left[\mu^2 \frac{\partial^2\varphi_n}{\partial x^2}+(n+2)(n+1)\varphi_{n+2}\right](z+1)^n = 0 \qquad (21)$$

Since z be an arbitrary value in $(-1, \varepsilon\eta)$, so each coefficient in power of (z + l) must be zero, thus

$$\varphi_{n+2} = -\frac{\mu^2}{(n+2)(n+1)}\frac{\partial^2\varphi_n}{\partial x^2}, \quad n = 0,1,2,..... \quad (22)$$

On the other hand, substituting Eq. (20) into Eq. (8) yields $\varphi_1 = 0$. Therefore, for all odds, $\varphi_n = 0$, i.e.,

$\varphi_1 = \varphi_3 = \varphi_5 = ... = 0.$

Supposing that $\varphi_0 = \Phi$, we have

$$\varphi_{2n} = (-1)^n \frac{\mu^{2n}}{(2n)!}\frac{\partial^{2n}\Phi}{\partial x^{2n}}, \qquad \varphi_{2n+1} = 0,\ n = 0,1,2,3..............$$

Now, the expression of velocity potential φ is obtained:

$$\varphi = \sum_{n=0}^{\infty}(-1)^n \frac{\partial^{2n}\Phi}{\partial x^{2n}}\frac{\mu^{2n}}{(2n)!}(z+1)^{2n}$$

By linear approximation, we also consider $\Phi = q_j \cos X$. (23)

Therefore, the velocity potential φ can be found to be: $\varphi(x,t,z) = q_j \cos X \cosh \mu(z+1).$

$$\varphi(x,z,t) = \Phi \cosh \mu(z+1), \qquad by \quad \text{using } (23)$$

Now

$$\frac{\xi^2}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2 + \frac{1}{\mu^2}\left(\frac{\partial\varphi}{\partial z}\right)^2\right] = -\frac{\xi^2}{4}q_j{}^2 \cos 2X + \frac{\xi^2}{4}q_j{}^2 \cos 2h\mu(z+1).$$

$$L=\int_{-1}^{\varepsilon\eta}\left\{\frac{\partial\varphi}{\partial t}+\frac{\varepsilon^2}{2}\left[\left(\frac{\partial\varphi}{\partial x}\right)^2+\frac{1}{\mu^2}\left(\frac{\partial\varphi}{\partial z}\right)^2\right]+z\right\}dz$$

$$=\int_{-1}^{\varepsilon\eta}\varepsilon\frac{\partial\phi}{\partial t}\cosh\mu(z+1)dz-\frac{\mu^2}{4}q_j^{\,2}\int_{-1}^{\varepsilon\eta}\cos 2Xdz+\frac{\varepsilon^2}{4}q_j^{\,2}\int_{-1}^{\varepsilon\eta}\cos 2h\mu(z+1)dz+\int_{-1}^{\varepsilon\eta}zdz$$

$$=\frac{\varepsilon}{\mu}\frac{\partial\phi}{\partial t}\sinh\mu(\varepsilon\eta+1)-\frac{\varepsilon^2}{4}q_j^{\,2}\cos 2X(1+\varepsilon\eta)+\frac{\varepsilon^2}{8\mu}q_j^{\,2}\sin 2h\mu(1+\varepsilon\eta)+\frac{1}{2}(\varepsilon\eta)^2-\frac{1}{2}.$$

Neglecting the constant term $\frac{1}{2}$, the above equation yields

$$L=\frac{\varepsilon}{\mu}\frac{\partial\Phi}{\partial t}\sinh\mu(1+\varepsilon\eta)-\frac{\varepsilon^2}{4}q_j^{\,2}\cos 2X(1+\varepsilon\eta)+\frac{\varepsilon^2}{8\mu}q_j^{\,2}\sinh 2\mu(1+\varepsilon\eta)+\frac{1}{2}(\varepsilon\eta)^2 \qquad (24)$$

The case of $\varepsilon=O(\mu^2)$, was considered by Benjamin(1967)and Whitham (1967) who obtained the Korteweg de Vries (KdV) equation. Here we also consider the case, and expand Lagrangian function up to $O(\mu^8)$order

$$L=\varepsilon\left(1-\frac{1}{6}\mu^2+\frac{19}{360}\mu^4-\frac{401}{630\times 24}\mu^6\right)q_j\sin X\left[1+\frac{\mu^2}{6}+\frac{\mu^4}{5!}+\frac{\mu^6}{7!}+\varepsilon\left(1+\frac{\mu^2}{2}+\frac{\mu^4}{24}\right)\eta+\frac{1}{2}\varepsilon^2\mu^2\eta^2\right]$$
$$+\varepsilon\mu^2q_j'\cos X\left[1+\frac{\mu^2}{6}+\frac{\mu^4}{120}+\varepsilon\left(1+\frac{\mu^2}{2}\right)\eta\right]-\frac{\varepsilon^2}{4}q_j^{\,2}\cos 2X(1+\varepsilon\eta)$$
$$+\frac{\varepsilon^2}{4}q_j^{\,2}\left[1+\frac{2}{3}\mu^2+\frac{2}{15}\mu^4+\varepsilon\eta+2\varepsilon\mu^2\eta+\frac{2}{3}\varepsilon\mu^4\eta\right]+\frac{1}{2}\varepsilon^2\eta^2+O(\mu^8) \qquad (25)$$

Hou et al.(1998) used the lowest-order of η in their article.

Let $\eta=a_1+a_2\mu^2+a_3\mu^4+a_4\mu^6+O(\mu^6)$

Substituting it into Eq. (25), we can see that we should expand η to μ^4,

$$\eta=a_1+a_2\mu^2+a_3\mu^4+O(\mu^6)$$

Based on the dynamical boundary condition of the free surface Eqn. (9), we have

$$\eta = -\frac{\partial \varphi}{\partial t}\bigg|_{z=\varepsilon\eta} - \frac{\varepsilon}{2}\left[\left(\frac{\partial \varphi}{\partial x}\right)^2 + \frac{1}{\mu^2}\left(\frac{\partial \varphi}{\partial z}\right)^2\right]\Bigg|_{z=\varepsilon\eta}$$

$$\therefore a_1 + a_2\mu^2 + a_3\mu^4 + O(\mu^6) = -q_j \sin X - \frac{\mu^2}{3} q_j \sin X - \mu^2 \dot{q}_j(\tau)\cos X - \frac{\varepsilon}{2} q_j^{\,2} \sin^2 X$$
$$+ \varepsilon\mu^2\left[q_j^{\,2}\sin^2 X - \frac{1}{2}q_j^{\,2}(\tau)\right] - \frac{\mu^4}{90} q_j \sin X - \frac{\mu^4}{2}\dot{q}_j \cos X + O(\mu^6) \qquad (26)$$

From Eq. (26), equating the coefficients of constant, μ^2 and μ^4 terms, we have

$$\begin{aligned} a_1 &= -q_j \sin X \\ a_2 &= -\dot{q}_j \cos X - \frac{1}{3} q_j \sin X - \frac{1}{2} q_j^{\,2} \sin^2 X \\ a_3 &= -\frac{1}{2}\dot{q}_j \cos X - \frac{1}{2} q_j^{\,2} + q_j^{\,2} \sin^2 X - \frac{1}{90} q_j \sin X \end{aligned} \qquad (27)$$

Eq.(1) can be rewritten as $\delta \int L\, d\tau = 0$

where $I[\dot{q}_j, q_j] = \frac{1}{2\pi}\int_0^{2\pi} L dX.$ (28)

From eq. (28) we have the Lagrangian

$$L(q_j, \dot{q}_j) = -\frac{1}{4}\varepsilon^2\mu^4\dot{q}_j^{\,2} - \frac{3}{64}\varepsilon^4 q_j^{\,4}. \qquad (29)$$

Obviously, Lagrangian is a function of generalized coordinates and generalized velocity.

Then from Lagrange's Equation of motion

$$\ddot{q}_j - \frac{3}{8}\frac{\varepsilon^2}{\mu^4} q_j^{\,3} = 0 \qquad (30)$$

Solving this for $\varepsilon = \mu^2$, $\quad q_j = \frac{4}{\sqrt{3}}\tau^{-1}$ (31)

From Eq. (31) it is obvious that q_j decreases with τ .Lagrange's equations use generalized velocities and

generalized coordinates where generalizedvelocities be $\dot{q}_j(\tau) = -\frac{4}{\sqrt{3}}\tau^{-2}$ (32)

which gives a wave model from Euler- Lagrangian equation. Thus the generalized velocity decreases with time. But at larger time generalized velocity will be diminished.

Conclusion

Firstly, we have derived the Lagrangian function which is expanded up to $O(\mu^8)$, then water wave equations are solved using Hamilton's principle. From Lagrange's equation of motion, it is seen that the generalized velocity decreases with time τ . Generalized velocity will be diminished when time is large. Again from the above discussion of non-dimensional free surface profiles, wave crest and trough will be flattened at larger time. The results obtained above can be studied where velocity of wave increases in deep water.

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