# COMPARATIVE STUDY OF THE SBA AND MOL METHODS. APPLICATION TO SOME SYSTEMS OF NONLINEAR PARTIAL DIFFERENTIAL EQUATIONS.



**Abstract**
In this paper, we have solved two systems of nonlinear partial differential equations(PDE) :the Fisher-Murray and Fitz Hugh-Nagumo. These systems of PDE are solved using the numerical Method of Lines(MOL)and the computed solution are compared with the ones obtained from SBA method(combination of Adomian method, Picard and successive approximation) by means of Matlab routines.

**Mots clés : MOL ; Systems of nonlinear PDE Fitz Hugh-Nagumo ; Systems of nonlinear PDE Fisher-Murray ; Matlab.**

**AMS classification codes** : 34A05, 34A08, 42A10, 40A30, 65B10

## 1 Introduction

System of nonlinear equations arise in many fields of applied mathematics and engineering. The methods to find their solutions are of fondamental importance. As analytical solution are rarely available, the research of efficient numerical methods are essential. Consider the following system of nonlinear partial differential equations ([10]; [12]; [13]) of type

$$\begin{cases} u_t &= k_1 u_{xx} + R\left(u, u_x, v, v_x\right), \ (x,t) \in \Omega \times [0,T] \\ v_t &= k_2 v_{xx} + Q\left(u, u_x, v, v_x\right), (x,t) \in \Omega \times [0,T] \end{cases} \tag{1.1}$$

where $u = u(x,t)$ and $v = v(x,t)$ are dependant variable of independant spatial variable $x \in \Omega = [0.L]$ and temporal $t \in [0,T]$, $R$ and $Q$ are the reaction nonlinear functions of $u$,$u_x$,$v$,$v_x$. The contants coefficients $k_1 > 0$ and $k_2 > 0$ are the thermal diffusibility of the media. For brievity $u_t = \dfrac{\partial u}{\partial t}$ and $u_{xx} = \dfrac{\partial^2 u}{\partial x^2}$. The system of equations (1.1) wil be solved on the spatial interval $[0,L]$ subject to boundary conditions for $u$

$$\begin{cases} u(0,t) &= a(t) \\ u_x\left(L,t\right) &= b(t) \end{cases} \tag{1.2}$$

and for $v$

$$\begin{cases} v(0,t) &= \alpha(t) \\ v_x\left(L,t\right) &= \beta(t), \ t \in [0,T] \end{cases} \tag{1.3}$$

and the initial conditions

$$\begin{cases} u(x,0) &= s(x) \\ v\left(x,0\right) &= r(x), \ x \in [0,L] \end{cases} \tag{1.4}$$

The boundary condiions (1.2) and (1.3) give the values of the two solution and their flux at the two ends of space domain as function of $t$. The equations in (1.4) specify the initial conditions.

We aim at solving numerically this problem, using the Method of Lines(MOL) ([1], [2], [3]; [4], [5]) and the SBA method ([24], [25], [26], [27])and compare their numerical solution. The paper is organized as follows. In section 2 we describe this method. In section 3, we apply these methods for solving two numerical example : the system of non linear equations of FitzHugh-Nagumo and the system of nonlinear equations of Fisher-Murray.

# 2 METHOD

We have the choice to discretize both in space and time to obtain a set of nonlinear algebraic equations(AEs) or to discretize only the space derivative and obtaining a set of non linear differential algebraic equations(DAEs). The last approach is retained in this work.The method of lines(MOL) ([6], [17], [22], [23], [24]) is a general way to convert a partial differential equation(PDE) [14]; [15] in the form of system of ordinary differential equations(ODE) see [11], [17] [20].[21] The derivatives with respect to the space variables in PDE are discretized to obtain a system of ODEs in time variable ([13]). A suitable ODE solver ([19] [20]) is used for the solution of ODE system. This method is give a very accurate numerical solution for linear and non linear PDE. We define a uniform mesh $0 = x_0 < x_1 < ... < x_N = L$ with

$$x_i = (i-1)\,h, \ i = 1,2,...,N, \ h = \frac{L}{N-1} \tag{2.1}$$

to approximate (1.1) along $x = x_i$ with

$$\left\{\begin{array}{rcl} u_t(x_i,t) & = & k_1 u_{xx}(x_i,t) + R\left(u(x_i,t), u_x(x_i,t), v(x_i,t), v_x(x_i,t)\right), \\ v_t(x_i,t) & = & k_2 v_{xx}(x_i,t) + Q\left(u(x_i,t), u_x(x_i,t), v(x_i,t), v_x(x_i,t)\right), i = 1,2,...,N-1 \end{array}\right. \tag{2.2}$$

Let $u_i(t) = u(x_i,t)$ and $v_i(t) = v(x_i,t)$. The equation (1.1) can be dsicretized on the uniform mesh (2.1), using the finite difference method [16] [17] [18] with the central difference approximation to obtain

$$\left\{\begin{array}{rcl} \frac{du_i}{dt}(t) & = & k_1 \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}}{h^2} + R_i \ , i = 1,2,...,N-1 \\ \frac{dv_i}{dt}(t) & = & k_2 \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h^2} + Q_i \ , i = 1,2,...,N-1 \end{array}\right. \tag{2.3}$$

where

$$R_i = R\left(u_i \ , \delta p_i, v_i, \delta q_i\right), Q_i = Q\left(u_i \ , \delta p_i, v_i, \delta q_i\right) \tag{2.4}$$

and

$$\delta p_i = \frac{u_{i+1} - u_{i-1}}{2h}, \delta q_i = \frac{v_{i+1} - v_{i-1}}{2h} \tag{2.5}$$

The boundary conditions (1.2) can be discretized to give

$$\left\{\begin{array}{rcl} u_0(t) & = & a(t) \\ u_{N+1} & = & u_{N-1} + 2hb(t) \end{array}\right. \tag{2.6}$$

and for $v$

$$\left\{\begin{array}{rcl} v_0(t) & = & \alpha(t) \\ v_{N+1} & = & v_{N-1} + 2h\beta(t) \end{array}\right. \tag{2.7}$$

For initial condition,the discretization of (1.4) give

$$\left\{\begin{array}{rcl} u_i \ (0) & = & s_i \ , 1 \leq i \leq N \\ v_i\,(0) & = & r_i, 1 \leq i \leq N \end{array}\right. \tag{2.8}$$

By introducing 2.6 and (2.7) in (2.3) by taking $i = 1$ and $i = N$, we get

$$\left\{\begin{array}{rcl} \frac{du_1}{dt}(t) & = & \frac{k_1}{h^2}\left(u_2(t) - 2u_i(t) + a(t)\right) + R_1 \\ \frac{dv_1}{dt}(t) & = & \frac{k_2}{h^2}\left(v_2(t) - 2v_1(t) + \alpha(t)\right) + Q_1 \end{array}\right. \tag{2.9}$$

and

$$\left\{\begin{array}{rcl} \frac{du_N}{dt}(t) & = & \frac{k_1}{h^2}\left(2u_N(t) + 2u_{N-1}(t) + 2hb(t)\right) + R_N \\ \frac{dv_N}{dt}(t) & = & \frac{k_2}{h^2}\left(2v_N(t) + 2v_{N-1}(t) + 2h\beta(t)\right) + Q_N \end{array}\right. \tag{2.10}$$

The equation

$$\left\{\begin{array}{rcl} \frac{du_i}{dt}(t) & = & k_1 \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}}{h^2} + R_i \ , i = 2,...,N-1 \\ \frac{dv_i}{dt}(t) & = & k_2 \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h^2} + Q_i \ , i = 2,...,N-1 \end{array}\right. \tag{2.11}$$

can be added to others in (2.9) and (2.10).

## 2.1 Vectoriel and matricial form

We let

$$\mathbf{w} \ = [u_1(t), u_2(t), ..., u_{N-1}(t), u_N(t), v_1(t), .v_2(t) ..., v_N(t)]^T \tag{2.12}$$

$$\mathbf{F}(\mathbf{w}\ ) = \begin{bmatrix} \frac{k_1}{h^2}(u_2(t) - 2u_i(t) + a(t)) + R_1 \\ ... \\ k_1 \frac{u_{i+1}(t) - 2u_i(t) + u_{i-1}}{h^2} + R_i \\ ... \\ \frac{k_1}{h^2}(2u_N(t) + 2u_{N-1}(t) + 2hb(t)) + R_N \\ \frac{k_2}{h^2}(v_2(t) - 2v_1(t) + \alpha(t)) + Q_1 \\ ... \\ k_2 \frac{v_{i+1}(t) - 2v_i(t) + v_{i-1}(t)}{h^2} + Q_i \\ ... \\ \frac{k_2}{h^2}(2v_N(t) + 2v_{N-1}(t) + 2h\beta(t)) + Q_N \end{bmatrix} \tag{2.13}$$

where $2 \leq i \leq N-1$. Using (2.8) as initial condition, and noting by

$$\mathbf{g} = [r_1(t), r_2(t), ..., r_{N-1}(t), r_N(t), s_1(t), .s_2(t) .., s_N(t)]^{\mathbf{T}} \tag{2.14}$$

the assciate vectoriel form we then obtain

$$\mathbf{w}(0) = \mathbf{g}, \tag{2.15}$$

The equations $(2.9)-(2.11)$ and $(2.15)$ give the following autonomous system of ordinary differential equation.

$$\begin{cases} \frac{d\mathbf{w}}{dt}(t) & = & \mathbf{F}(\mathbf{w}(t)),\ t > 0 \\ \mathbf{w}(0) & = & \mathbf{g} \end{cases} \tag{2.16}$$

The MOL approximation replaces a PDE system in (1.1) with an initial-value ODE system in (2.16).This ODE system is integrated using a standard routine. In this way, the solution take avantage of the progress in ODE numerical integrators available in Matlab like RK4 or ode15s, ode 23tb, ... for stiff system of ODE.

# 3 NUMERICAL EXPERIMENTS

In this section, we solve some examples to show the efficiency of the method of lines and compare the resulting numerical solution with the one obtained by the SBA method.

**Exemple 3.1** *The first example consider the FITZ Hugh-Nagumo sytem of equations* $(1.1)-(1.2)$

$$\begin{cases} u_t &= \zeta u_{xx} + 2u + v, \\ v_t &= \zeta v_{xx} + uv + v^2 \end{cases}$$

where $L = \pi, T = 10,\ k_1 = k_2 = \zeta\ R(u, v) = 2u + v,\ Q(u, v) = uv + v^2$ with initial conditions

$$\begin{cases} u(x, 0) &= \theta_1 \cos x + \theta_2 \sin x \\ v(x, 0) &= -(\theta_1 \cos x + \theta_2 \sin x) \end{cases}$$

and boundary conditions

$$\begin{cases} u(0, t) &= |\theta_1| \exp(|1 - \zeta|\, t) \\ v(0, t) &= -\ |\theta_1| \exp(|1 - \zeta|\, t) \end{cases}$$

and

$$\begin{cases} u(\pi, t) &= -|\theta_1| \exp(|1 - \zeta|\, t) \\ v(\pi, t) &= |\theta_1| \exp(|1 - \zeta|\, t) \end{cases}$$

$$\begin{cases} u(\pi, t) &= -|\theta_1| \exp(|1 - \zeta|\, t) \\ v(\pi, t) &= |\theta_1| \exp(|1 - \zeta|\, t) \end{cases}$$

The parameters $\theta_1 = .05\,;\ \theta_2 = 1.5\,;\ \zeta = 0.5$ are considered for the demonstration The solutions give by the application of SBA method ([24], [25], [26], [27]) give $u(x, t) = |\theta_1 \cos x + \theta_2 \sin x| \exp(|1 - \zeta|\, t)$ and $v(x, t) = -|\theta_1 \cos x + \theta_2 \sin x| \exp(|1 - \zeta|\, t)$ wich can be used for the comparison of our method.



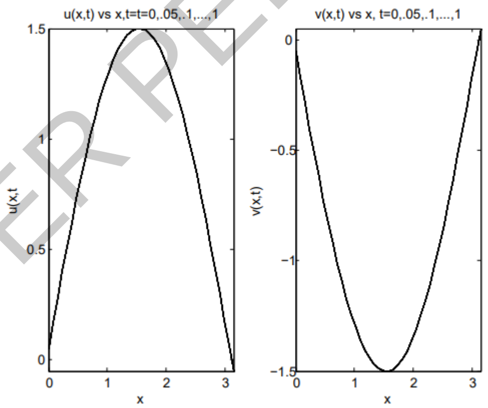


Figure 1 – MOL Solution for u and v at a fixed time

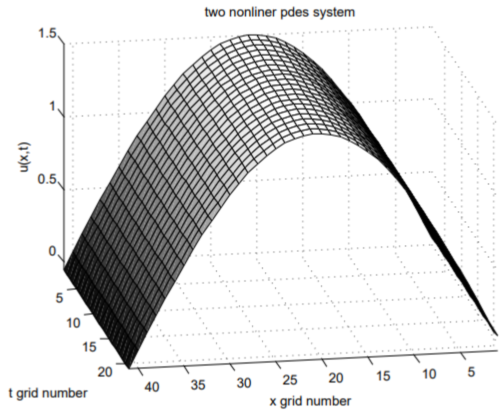


FIGURE 2 – MOL Solution for u

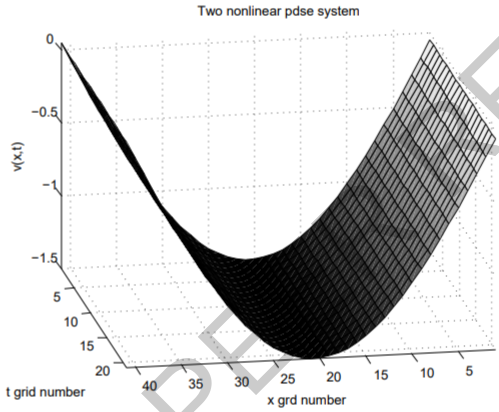


FIGURE 3 – MOL Solution for v

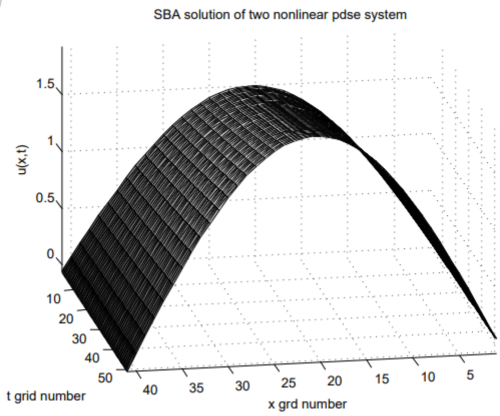


FIGURE 4 – SBA Solution for u

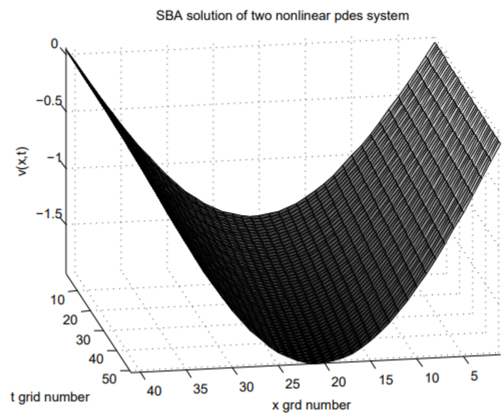


FIGURE 5 – SBA Solution for v

**Exemple 3.2** *The second example consider the sytem of equations* (1.1) – (1.2) *de type Fisher-Murray*

$$\begin{cases} u_t &= u_{xx} + [u.u_x]_x + u^2 - v^2, \\ v_t &= v_{xx} - v_x^2 + u^2 \end{cases}$$

With $L = \pi, T = 10$, $k_1 = k_2 = 1$, $R(u, u_x, v, v_x) = [u.u_x]_x + u^2 - v^2$, $Q(u, u_x, v, v_x) = -v_x^2 + u^2$ with initial conditions

$$\begin{cases} u(x, 0) &= \sin x \\ v(x, 0) &= \cos x \end{cases}$$

and boundary conditions

$$\begin{cases} u(0, t) &= 0 \\ v(0, t) &= \exp(-t) \end{cases}$$

and

$$\begin{cases} \dfrac{\partial u}{\partial x}(\pi, t) &= -\exp(-t) \\ \dfrac{\partial v}{\partial x}(\pi, t) &= 0 \end{cases}$$

The semi-analytical solution computed using SBA method is given by $u(x, t) = \sin x \exp(-t)$ and $v(x, t) = \cos x \exp(-t)$ to be compared with MOL numerical method. The different numerical analysis for MOL method has been undertaken by dividing the spatial domain $\Omega = [0, \pi]$, using N=101 with $h = \dfrac{\pi}{N-1}$ and replacing derivatives using the finite difference method for order two. For resulting ODE, we ressort to ODE solver ode15s wich is convenient for stiff problem in the interval $[0, .1]$. The comparison was made by confronting the graph of the results provide by MOL method and SBA method.

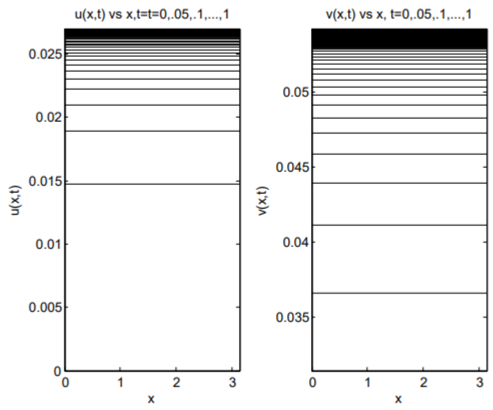


FIGURE 6 – MOL Solution for u and v at a fixed time

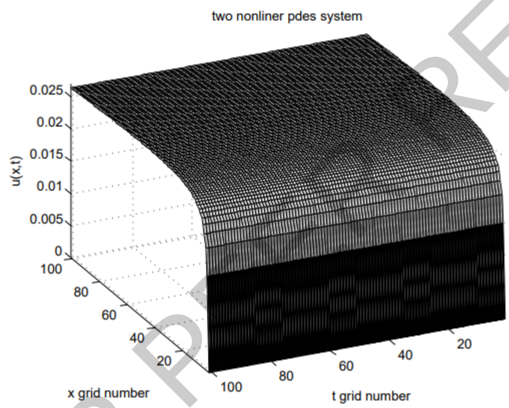


FIGURE 7 – MOL Solution for u

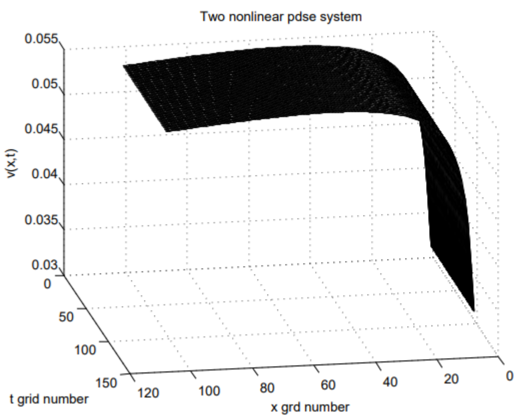


FIGURE 8 – MOL Solution for v

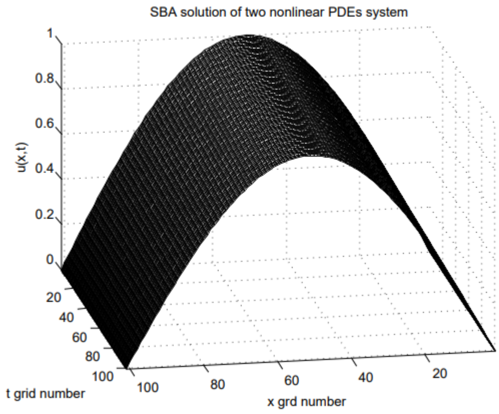


FIGURE 9 – SBA Solution for u

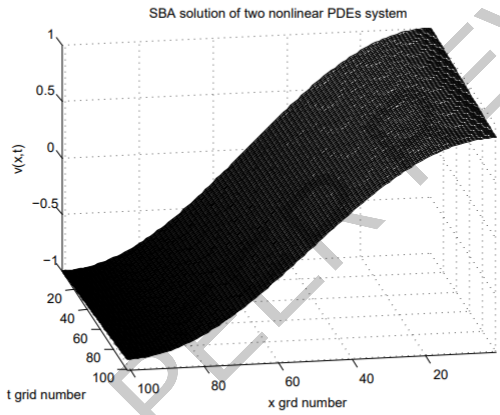


FIGURE 10 – SBA Solution for v



# 4 Conclusion

This paper investigated MOL method for solving the one-dimensional systems of nonlinear partial differential equations and compared the resulting solution of another semi-analytical SBA method. The method of MOL proceeds in two separate steps. Firstly, spatial derivatives are replaced with finite difference, finite element , finite volume, spectral method and the resulting system of ordinary differential equations is integrated over time. The availability of high-quality numerical algorithm for solution of stiff system of odes facilitated the computation of the desired results. The method is tested on two system of nonlinear partial differential equations, namely the FITZ Hugh-Nagumo and Fisher-Murray systems of nonlinear equations and the computational solution confirmed the efficiency when compared with the solution obtained by applying the sem-ianalytical method SBA.

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