

CONSTRAINED STOCHASTIC INVENTORY CONTROL MODELS FOR MULTI-ITEM WITH VARIABLE DEMAND

Abstract

Constrained multi-item inventory system with variable demands are considered. The demand rates of five selected multi-item - Cowbell, Milo, SMA, Cerelac and Golden Morn were modeled as Weibull, Normal and Lognormal probability distributions respectively with the aid of chi-squared multinomial goodness-of-fit test. The respective probability distributions with estimated location parameters: 491.55, 536.92, 10.5, 2.1926 and 5.3103 were used as the basis of probabilistic inventory models to obtain dynamic EOQ for each item under each constraint subject to Kuhn, Karush and Tucker (KKT) conditions as against the use of simple averages in deterministic inventory models. The optimal values of these constraints: available warehouse space (124sq.ft), specified level of inventory (94 units), limited capital (76,671.52 naira) and number of orders (1/month) were obtained using the optimal EOQ values to establish optimal inventory level and constraints level/capacity for each item in order to avoid shortage or excess stock.

Keywords: Constrained inventory model, multi-item, variable demand, optimal inventory, probability distribution, variable demand.

1. Introduction

Possessing a high amount of inventory for a long period of time is not usually good for business because of inventory storage, obsolescence and spoilage costs. However, possessing too little inventory isn't good either, because the business runs the risk of losing out on potential sales and potential market share as well. The economic order quantity (EOQ) assumes that demand occurs at known constant rate and supply fulfill the replenishment order after a fixed lead time. Unfortunately, the real world is not as ideal as that. In reality, demand rate is rarely constant and hard-to-predict market is common in most practical situations. Therefore, inventory management forecasts and strategies, such as a just-in-time inventory system can be deterministic or probabilistic. According to [1] probabilistic EOQ model is an inventory model that is close to the real situation that retailers face because demand will vary from time to time. This probabilistic inventory model will incorporate the variation of the demand and uncertain lead time. Demand variation will cause a shortage especially during lead time when retailer only has a limited amount of goods to cover the demand during lead time and the goods ordered have not arrived yet. Based on that situation, there are three possibilities that can happen to the probabilistic

inventory model: The first one is when demand during lead time is constant but the lead time itself varies. The second is when lead time is constant but demand during lead time varies, and the last possibility is when both lead time and demand during lead time vary which is the case in this study. On this perspective, a multi-objective inventory model of deteriorating items with stock dependent demand under limited imprecise storage area and total cost budget was formulated by [2] to handle stock-demand control of deteriorating items, while a proposed multi-item inventory control model with space capacity constraint for pharmaceutical products was considered by [3]. In a dynamic single stage multi-item inventory control model by [4], the average holding cost and stock out probabilities for the components were determined considering a given service level for customers' demands and lead time uncertainties. Further studies by [5] and [6] have examined inventory policies for multiple substitutable items in case of stock out with stochastic demand, fixed ordering costs and constant holding costs in effort to formalize the process of maintaining optimal inventory.

The perspective of probabilistic continuous review inventory models with constant units of cost and lead-time demand as a random variable was presented by [7]. He gave heuristic approximate treatment for each of the backorders and the lost sales cases, while [8] studied the probabilistic single-item, single source (SISS) inventory system with zero lead-time, using classical optimization. Also, [9] considered probabilistic price dependent demand and imprecise goal and constraints. The objective was to obtain a multi-item inventory model with stochastic price-dependent demand which probability distribution depends on selling price as a parameters. Similarly, Authors in [10] treated multi-item inventory system with budgetary constraint comparison between the Lagrange and the fixed cycle approach under the Kuhn, Karush and Tucker (KKT) conditions. These conditions were originally named after Harold W. Kuhn and Albert W. Tucker, who first published the conditions in 1951 but was later discovered that the necessary conditions for this problem had been stated by William Karush in his Master's thesis in 1939. In a later development, [11] considered both deterministic and probabilistic versions of power demand patterns with a variable rate of deterioration, while [12] considered two types of holding cost variation: (a) a nonlinear function of storage time and (b) a nonlinear function of storage level. According to [13], inventory control problems in real world usually involve multiple products which are often necessary for inventory holding thousands of items. Also, they examined the difficulties encountered in the practice of inventory control

management and concluded that a large gap exists between theory and practice in inventory management. In [14], a multi-item probabilistic inventory model that considered expiration factor, all unit discount policy and warehouse capacity constraints was considered. The characteristics involved in this study were probabilistic demand, perishable products, and warehouse constraints for multi-item inventory models. These conditions occur in several industries that consider perishable factors and warehouse constraints (example are companies that produce food, food sales agents, and retail goods to end customers, among others). The Karush-Kuhn-Tucker condition approach was used to solve the warehouse capacity problem to find the optimum point of a constrained function. The results yielded two optimal ordering times, namely ordering time-based on warehouse capacity and joint order time.

2. Related works

A realistic and general single period for multi-item with budgetary and floor or shelf space constraints, where demand of item follow uniform probability distribution was developed by [15]. Also, [16] developed a multi-item inventory control model with instantaneous supply where demand is deterministic and follows uniform distribution for perishable items. The use of KKT conditions was also employed by [17] to solve a multi-item inventory model with shortages and demand dependent on unit cost with storage space and set up cost constraints. The cost parameters were treated as fuzzy variables because of its imprecise nature. Also, [18] developed an inventory model for deteriorating and ameliorating items with capacity constraint for storage facility. Other factors that were also considered in the model were the effect of inflation and time value of money in the profit as well as cost parameter and associated profit. In another development a multi-item multi-period inventory control model for known-deterministic variable subject to limited available budget was formulated by [19] which considered shortages in combination with backorder, unit discount and lost sales. The model was formulated into a fuzzy multi-criteria decision making (FMCDM) framework represented as a mixed integer nonlinear programming problem with the objectives to minimize both the total inventory cost and the required storage space.

Furthermore, [20] proposed a new general probabilistic multi-item, single source inventory model with varying mixture shortage cost under restrictions on backorder cost and expected varying lost sales cost. In [1] individual and joint replenishment policies which consisted of

several products where the demands for these products followed Gamma distribution were formulated. The objective was to determine the optimal ordering quantity that minimizes the total cost for each product, and [21] examined the probability distributions of variable demand rate of multi-item inventory problem. The result showed that demand of selected products follow certain probability distributions namely; normal, uniform and Weibull distributions. Optimal order quantities and the probabilities of shortage and no shortage were also obtained for the selected products. The multi objective optimization method was utilized by [22] to solve a multi item inventory control model which was developed to optimize the total inventory cost and inventory layout management using a meta-heuristic algorithm named multi-objective particle swarm optimization (MOPSO) algorithm. Furthermore, [23] used the normal and exponential distributions in formulating probabilistic multi-item inventory models to minimize the expected total cost. Varying mixture of shortage cost (backorders and lost sales) were considered under certain constraints.

In [24], a deterministic multi-item inventory model where the total variable cost (TVC) was used as the objective function subject to a number of fixed constraints limitations/capacities was optimized under the KKT conditions to obtain optimal EOQ values for the items. We note that the average demand of items in [24] were obtained by the use of simple arithmetic mean formula over the period for the deterministic inventory model which does not reflect the dynamic behavior of the demand rate of the items, since demand is a variable. On this perspective, this work sought to convert the deterministic inventory model to constrained stochastic multi-item inventory model by modeling the variable demand rate using appropriate probability distribution functions. The identified probability distribution functions shall be used as the basis to obtaining dynamic EOQ models and associated parameters estimates for items subject to constraints under the KKT conditions.

1.2 Problem definition and Assumptions

A distributor company of multi-item gets its supply from different manufacturing companies at different times by placing an order to the manufacturing companies. The company is faced with several restrictions that limits her capacity to make orders at will and convenience. These restrictions (constraints) include total available warehouse space of 650sq.ft and total limited capital of 310,000 naira among others. However, only five items were selected out of various

multi-items in the warehouse for study with warehouse space, required capital, level of inventory and number of orders constraints as unknown. It therefore requires a robust and dynamic approach of estimating the demand of each of the selected item to enable her determine the ordering frequency and the best quantity to be ordered for each product at appropriate time to avoid shortage or excess stock within the sphere of the stated conditions (constraints). The capacities/levels of the unknown constraints for the selected items need be obtained to determine what quantities are left for the other items not considered in the study. Under this consideration, demand rate of an item is assumed to follow a probability distribution. The probability of lead time is between zero and one; $\text{Pr}[0 \leq LT \leq 1]$, shortages are not allowed, purchase cost and reordering costs do vary with time for a specified period.

2. Methodology

2.1 Formulation of EOQ Model with constraints

The following notations would be used in this work, except otherwise stated:

TVC = Total variable cost

n = total number of items being controlled simultaneously

f_i = floor area (storage space) required per unit of item i ($i = 1, 2, \dots, n$)

W = warehouse space limit to store all items in the inventory.

λ = non-negative Lagrange multiplier

D_i = annual demand for i th item

\bar{D}_i = average demand for each item ($i = 1, 2, 3, \dots, n$)

Q_i = order quantity for items i in inventory ($i = 1, 2, \dots, n$)

M = upper limit of average number of units for all items in the stock

C_i = price per unit of items in the stock

F = investment limit for all items in the inventory

C_{oi} = order cost per order ($i = 1, 2, 3, \dots, n$)

C_{hi} = cost of carrying one unit of an item in the inventory for a given length of time

A = Maximum value of order

B_i = Total quantity ordered for each item ($i = 1, 2, 3, \dots, n$)

S = Maximum shortage quantity

Let the decision variables be D_i, Q_i, C_{oi} and C_{hi} . The objective function as adapted from [20] is:

$$\text{Min TVC} = \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right] \quad (1)$$

The constraints:

1. Warehouse space availability:

$$\sum_{i=0}^n f_i Q_i \leq W \quad (2)$$

2. Capital limited:

$$\sum_{i=0}^n C_i Q_i \leq F \quad (3)$$

3. Inventory level specification:

$$\frac{1}{2} \sum_{i=0}^n Q_i \leq M \quad (4)$$

4. Order quantities:

$$\sum_{i=0}^n \frac{B_i}{Q_i} \leq A \quad (5)$$

5. Non negativity:

$$D_i, Q_i, C_{oi} \text{ and } C_{hi} \geq 0$$

The constrained inventory models are:

1. $\text{Min TVC} = \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right]$

Subject to: $\sum_{i=0}^n f_i Q_i \leq W$

2. $\text{Min TVC} = \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right]$

Subject to: $\sum_{i=0}^n C_i Q_i \leq F$

3. $\text{Min TVC} = \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right]$

Subject to: $\frac{1}{2} \sum_{i=0}^n Q_i \leq M$

$$4. \text{ Min TVC} = \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right]$$

$$\text{Subject to: } \sum_{i=0}^n \frac{B_i}{Q_i} \leq A$$

$$D_i, Q_i, C_{oi} \text{ and } C_{hi} \geq 0$$

By applying the KKT necessary and sufficient condition for optimal value of TVC, we obtain the following formulations with respect to each constraint:

1. **Optimization of TVC subject to warehouse space available constraint:**

$$\text{Min TVC} = \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right]$$

$$\text{Subject to: } \sum_{i=0}^n f_i Q_i \leq W$$

If the warehouse space required for each unit of item, i is f_i ($i=1,2, \dots, n$), then the total storage area (or volume) required by all n inventory items must be less than or equal to the total available space storage area (or volume) of the warehouse. This constraint indicates that even if all items reach their maximum inventory levels simultaneously, the warehouse space should be sufficient to store the inventory of these items with an assumption that all the five items are received together. Thus, the problem is to minimize the total variable inventory cost for each item under warehouse constraint.

$$\text{Let the Lagrange function be: } L(Q_i, \lambda) = \text{TVC} + \lambda[\sum_{i=1}^n f_i Q_i - W] \quad (6)$$

$$= \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right] + \lambda[\sum_{i=1}^n f_i Q_i - W]$$

The necessary condition for L to be minimum with respect to Q_i is:

$$\frac{\partial L}{\partial Q_i} = \frac{\partial}{\partial Q_i} \left[\sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right] + \lambda[\sum_{i=1}^n f_i Q_i - W] \right] = 0$$

$$= -\frac{2D_i C_{oi}}{Q_i^2} + \frac{C_{hi}}{2} + \lambda f_i = 0$$

$$\frac{-2D_i C_{oi} + Q_i^2 C_{hi} + 2\lambda f_i Q_i^2}{2Q_i^2} = 0$$

$$-2D_i C_{oi} = -Q_i^2 [C_{hi} + 2\lambda f_i]$$

$$Q_i^2 = \frac{2D_i C_{oi}}{C_{hi} + 2\lambda f_i}$$

$$Q_i^* = \sqrt{\frac{2D_i C_{oi}}{C_{hi} + 2\lambda f_i}}; \quad i = 1, 2, \dots, n \quad (7)$$

Also, the necessary condition for L to be minimum with respect to λ is:

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^n f_i Q_i - W = 0$$

$$\sum_{i=1}^n f_i Q_i = W ; \text{ since } \lambda \geq 0 \quad (8)$$

2. Optimization of TVC subject to limited capital constraint:

$$\text{Min TVC} = \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right]$$

$$\text{Subject to: } \sum_{i=0}^n C_i Q_i \leq F$$

Since investment on inventory is substantial for many organizations, decision makers must put a restriction on the amount of inventory to be carried. The inventory control policy is accordingly adjusted to achieve the objective of keeping total investment required within limit. Hence, the problem is to minimize the total variable inventory cost for each item under the investment constraint, along with the assumption that both demand and lead time are constant and known.

$$\text{Let the Lagrange function be: } L(Q_i, \lambda) = \text{TVC} + \lambda[\sum_{i=1}^n C_i Q_i - F] \quad (9)$$

$$= \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right] + \lambda[\sum_{i=1}^n C_i Q_i - F]$$

The necessary condition for L to be minimum with respect to Q_i is:

$$\frac{\partial L}{\partial Q_i} = \frac{\partial}{\partial Q_i} \left[\sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right] + \lambda[\sum_{i=1}^n C_i Q_i - F] \right] = 0$$

$$= -\frac{D_i C_{oi}}{Q_i^2} + \frac{C_{hi}}{2} + \lambda C_{oi} = 0$$

$$-2D_i C_{oi} + Q_i^2 C_{hi} + 2\lambda Q_i^2 C_{oi} = 0$$

$$Q_i^2 = \frac{2D_i C_{oi}}{C_{hi} + 2\lambda C_{oi}}$$

$$Q_i^* = \sqrt{\frac{2D_i C_{oi}}{C_{hi} + 2\lambda C_{oi}}}; \quad i = 1, 2, \dots, n \quad (10)$$

Also, the necessary condition for L to be minimum with respect to λ is:

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^n C_i Q_i - F = 0$$

$$\sum_{i=1}^n C_i Q_i = F; \text{ since } \lambda \geq 0 \quad (11)$$

3. Optimization of TVC subject to inventory level specification constraint:

$$\text{Min TVC} = \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right]$$

$$\text{Subject to: } \frac{1}{2} \sum_{i=0}^n Q_i \leq M$$

Since the average number of units in the inventory of an item, i is $Q_i/2$, and it is required that the average number of units of individual items held together in the inventory should not exceed the pre-specified number, M . The problem is to minimize the total variable inventory cost subject to the limitation of total average inventory level of items. Thus,

$$\text{Let the Lagrange function be: } L(Q_i, \lambda) = \text{TVC} + \lambda \left[\frac{1}{2} \sum_{i=1}^n Q_i - M \right] \quad (12)$$

$$= \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right] + \lambda \left[\frac{1}{2} \sum_{i=1}^n Q_i - M \right]$$

The necessary condition for L to be minimum with respect to Q_i is:

$$\frac{\partial L}{\partial Q_i} = \frac{\partial}{\partial Q_i} \left[\sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right] + \lambda \left[\frac{1}{2} \sum_{i=0}^n Q_i - M \right] \right] = 0$$

$$= -\frac{D_i C_{oi}}{Q_i^2} + \frac{C_{hi}}{2} + \frac{\lambda}{2} = 0$$

$$-2D_i C_{oi} + Q_i^2 [C_{hi} + \lambda] = 0$$

$$Q_i^2 = \frac{2D_i C_{oi}}{C_{hi} + \lambda}$$

$$Q_i^* = \sqrt{\frac{2D_i C_{oi}}{C_{hi} + \lambda}}; \quad i = 1, 2, \dots, n \quad (13)$$

Also, the necessary condition for L to be minimum with respect to λ is:

$$\frac{\partial L}{\partial \lambda} = \frac{1}{2} \sum_{i=1}^n Q_i - M = 0$$

$$\frac{1}{2} \sum_{i=1}^n Q_i = M; \quad \text{since } \lambda \geq 0 \quad (14)$$

4. Optimization of TVC subject to order quantity constraint:

$$\text{Min TVC} = \sum_{i=0}^n \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi} \right]$$

$$\text{Subject to: } \sum_{i=0}^n \frac{B_i}{Q_i} \leq A$$

Number of orders is very important in inventory as it eliminates the existence of shortage and excess. This additional constraint helps us to determine the minimum number of monthly orders where shortages is zero. For instance, a TVC was formulated in [25] by incorporating purchase with shortage and production with shortage. Contrary, our aim is to obtain the number of inventory orders that should be considered. This number of orders constraint will help us determine how many times or batches should items be ordered so as to minimize cost. The required EOQ model when the constant number of orders is active is obtained as follows:

$$\text{Let the Lagrange function be: } L(Q_i, \lambda) = \text{TVC} + \lambda \left[\sum_{i=0}^n \frac{B_i}{Q_i} - A \right] \quad (15)$$

$$= \frac{C_{oi}D_i}{Q_i} + \frac{C_{hi}(Q_i-s)^2}{2Q_i} + P_i \frac{S^2}{2Q_i} + \lambda \left[\sum_{i=0}^n \frac{B_i}{Q_i} - A \right]$$

The necessary condition for L to be minimum with respect to Q_i is:

$$\frac{\partial L}{\partial Q_i} = \frac{\partial}{\partial Q_i} \left[-\frac{C_{oi}D_i}{Q_i^2} + \frac{C_{hi}}{2} - \lambda \frac{B_i}{Q_i^2} \right] = 0, \quad s = 0$$

$$-C_{oi}D_i + \frac{Q_i^2 C_{hi}}{2} - \lambda B_i = 0$$

$$-2C_{oi}D_i + Q_i^2 C_{hi} - 2\lambda B_i = 0$$

$$Q_i^2 C_{hi} = 2C_{oi}D_i + 2\lambda B_i$$

$$Q_i^2 = \frac{2C_{oi}D_i + 2\lambda B_i}{C_{hi}}$$

$$Q_i^* = \sqrt{\frac{2C_{oi}D_i + 2\lambda B_i}{C_{hi}}}; \quad i = 1, 2, \dots, n \quad (16)$$

Also, the necessary condition for L to be minimum with respect to λ is:

$$\frac{\partial L}{\partial \lambda} = \sum_{i=0}^n \frac{B_i}{Q_i} - A = 0$$

$$\sum_{i=0}^n \frac{B_i}{\bar{D}_i} = A \quad (\text{where } Q_i \approx \bar{D}_i); \quad \text{since } \lambda \geq 0 \quad (17)$$

2.2 Probability distribution of demand rate of items

In reality, demand will vary from time to time. Thus, probabilistic inventory model will incorporate the variation of demand and uncertain lead time. In this case, demand of an item is not known to be deterministic but are considered variables. Hence, the need to obtain the probability distributions for the demand of the multi-item. The following cases were considered after necessary preliminary analysis:

1. When Demand of item is assumed to follow normal distribution.
2. When demand of item is assumed to follow Weibull distribution.
3. When demand of item is assumed to follow lognormal distribution.

2.2.1 Probability inventory model when demand follows a normal distribution

The probability distribution of the demand of the products were not known and were assumed to be normally distributed with the density function:

$$f_D(D, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(D-\mu)^2}{2\sigma^2}} ; D \geq 0, \sigma > 0 \quad (18)$$

Where D is the demand (random variable), μ and σ^2 are mean and variance of the normal distribution.

Estimation of parameters of normal distribution for quantity demand using Maximum Likelihood method

Let $L(\mu, \sigma^2; D_1, D_2, \dots, D_n)$ and $\ln(L f(\mu, \sigma^2; D_1, D_2, \dots, D_n))$ be the likelihood and loglikelihood functions of the normal distribution.

By substituting (18) in the loglikelihood function, we have:

$$\ln \prod_{i=1}^n \left[(2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(D_i - \mu)^2\right) \right] = \ln \left[(2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (D_i - \mu)^2\right) \right]$$

$$\ln(2\pi\sigma^2)^{-\frac{n}{2}} - \frac{1}{2\sigma^2} \sum_{i=1}^n (D_i - \mu)^2 = -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (D_i - \mu)^2$$

By minimizing the loglikelihood function with respect to μ , we have:

$$\frac{\partial L(\mu, \sigma^2; D_1, D_2, \dots, D_n)}{\partial \mu} = \frac{1}{\sigma^2} \sum_{i=1}^n (D_i - \mu) = 0$$

$$\sum_{i=1}^n D_i - n\mu = 0$$

$$\therefore \hat{\mu} = \frac{1}{n} \sum_{i=1}^n D_i \quad (19)$$

Also, by minimizing the loglikelihood function with respect to σ^2 , we have:

$$\frac{\partial \ln L(\mu, \sigma^2; D_1, D_2, \dots, D_n)}{\partial \sigma^2} = -\frac{n}{2} \left(\frac{2\pi}{2\pi\sigma^2} \right) + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^n (D_i - \mu)^2 = 0$$

$$-\frac{2\pi n}{4\pi\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^n (D_i - \mu)^2 = -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^n (D_i - \mu)^2 = 0$$

$$\begin{aligned}
-n + \frac{1}{\sigma^2} \sum_{i=1}^n (D_i - \mu)^2 &= -\sigma^2 + \frac{1}{n} \sum_{i=1}^n (D_i - \mu)^2 = 0 \\
\therefore \hat{\sigma}^2 &= \frac{1}{n} \sum_{i=1}^n (D_i - \mu)^2
\end{aligned} \tag{20}$$

2.2.2 Probability inventory model for demand with lognormal distribution

A continuous random variable D is said to have a lognormal distribution with mean μ and variance σ^2 if the density function is given by:

$$f_D(\ln(D), \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{(\ln(D)-\mu)^2}{2\sigma^2}} ; D_i \geq 0, \sigma > 0 \tag{21}$$

Where D is the quantity demand (random variable), μ and σ^2 are the mean and variance of the quantity demand.

Parameters Estimation of lognormal distribution

We use the maximum likelihood method of parameter estimation as follows:

Let $L(\mu, \sigma^2; \ln(D_1), \ln(D_2), \dots, \ln(D_n))$ and $\ln L[\mu, \sigma^2; \ln(D_1), \ln(D_2), \dots, \ln(D_n)]$ be the likelihood and log likelihood functions.

By substituting (21) in the loglikelihood function, we have:

$$\begin{aligned}
\ln \left[(2\pi\sigma^2)^{-\frac{n}{2}} \exp \left(-\frac{1}{2\sigma^2} \sum_{i=1}^n (\ln D_i - \mu)^2 \right) \right] &= \ln(2\pi\sigma^2)^{-\frac{n}{2}} - \frac{1}{2\sigma^2} \sum_{i=1}^n (\ln D_i - \mu)^2 \\
&= -\frac{n}{2} \ln(2\pi\sigma^2) - \frac{1}{2\sigma^2} \sum_{i=1}^n (\ln D_i - \mu)^2
\end{aligned}$$

By minimizing the loglikelihood function with respect to μ , we have:

$$\begin{aligned}
\frac{\partial L(\mu, \sigma^2; D_1, D_2, \dots, D_n)}{\partial \mu} &= \frac{1}{\sigma^2} \sum_{i=1}^n (\ln D_i - \mu)^2 = 0 \\
\sum_{i=1}^n \ln D_i - n\mu &= 0 \\
\therefore \hat{\mu} &= \frac{1}{n} \sum_{i=1}^n \ln D_i
\end{aligned} \tag{22}$$

Also by minimizing the loglikelihood function with respect to σ^2 , we have:

$$\begin{aligned}
\frac{\partial L(\mu, \sigma^2; \ln(D_1), \ln(D_2), \dots, \ln(D_n))}{\partial \sigma^2} &= -\frac{n}{2} \left(\frac{2\pi}{2\pi\sigma^2} \right) + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^n \ln(D_i - \mu)^2 = 0 \\
-\frac{2\pi n}{4\pi\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^n [\ln(D_i) - \mu]^2 &= -\frac{n}{2\sigma^2} + \frac{1}{2(\sigma^2)^2} \sum_{i=1}^n [\ln(D_i) - \mu]^2 = 0
\end{aligned}$$

$$-n + \frac{1}{\sigma^2} \sum_{i=1}^n [\ln(D_i) - \mu]^2 = -\sigma^2 + \frac{1}{n} \sum_{i=1}^n [\ln(D_i) - \mu]^2 = 0$$

$$\therefore \hat{\sigma}^2 = \frac{1}{n} \sum_{i=1}^n [\ln(D_i) - \mu]^2 \quad (23)$$

2.2.3 Probability inventory model for demand with Weibull distribution

The probability density function of a two parameter Weibull random variable, D is;

$$f(D, \alpha, \beta) = \begin{cases} \left(\frac{\alpha}{\beta}\right) (D)^{\alpha-1} e^{-\left[\frac{D}{\beta}\right]^\alpha} & D > 0 \\ 0, & \text{if } D \leq 0 \end{cases} \quad (24)$$

Estimation of parameters

By use of the least squares method of estimation we obtained the following density and distribution functions respectively:

$$f(D, \alpha, \beta) = \left(\frac{\alpha}{\beta}\right) D^{\alpha-1} \prod_{i=1}^n \left(\frac{D}{\beta}\right)^{\alpha-1} e^{\sum_{i=1}^n \left[\frac{D}{\beta}\right]^\alpha}$$

$$F(D, \alpha, \beta) = 1 - e^{\left[\frac{D}{\beta}\right]^\alpha} \quad (25)$$

The log function of the distribution function is:

$$\ln \left[\frac{1}{1-F(D)} \right] = \alpha \ln D - \alpha \ln \beta \quad (26)$$

Let D_i be a random sample of the demand and $F(D)$ is estimated by the median rank method and replaced as follows:

$$F(D) = \frac{i-0.3}{n+0.4}, (D_i, i = 1, 2, \dots, n) \text{ and } (D_1 < D_2 < \dots < D_n)$$

Eq (26) becomes;

$$Y = \alpha X + \lambda \quad (27)$$

Where $Y = \ln[-\ln(1 - F(D))]$

$$X_i = \ln D$$

and $\lambda = -\alpha \ln \beta$

Estimates of α, λ and β can be obtained in (27) as:

$$\hat{\alpha} = \frac{\left[\sum_{i=1}^n XY - \frac{1}{n} \sum_{i=1}^n X \sum_{i=1}^n Y \right]}{\sum_{i=1}^n X^2 - \frac{1}{n} (\sum_{i=1}^n X)^2}$$

$$\hat{\lambda} = \frac{1}{n} \sum_{i=1}^n Y - \frac{\alpha}{n} \sum_{i=1}^n X$$

$$\hat{\lambda} = \bar{Y} - \hat{\alpha} \bar{X}$$

$$\hat{\beta} = e^{-\frac{\lambda}{\alpha}}$$

The mean of Weibull distribution is given by;

$$E[D] = \beta \Gamma\left(\alpha + \frac{1}{\alpha}\right) \quad (28)$$

2.3 Chi-square goodness-of-fit test

The chi-square goodness-of-fit test would be used to determine how well the sample data fits a distribution from a population. It establishes the discrepancy between the observed values and expected values.

The test statistic is given by:

$$\chi^2 = \sum_{i=1}^n \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \sim \chi_{\alpha; n-1}^2 \quad (29)$$

Where O_{ij} are the observed value in cell (i, j)

E_{ij} are the expected value in cell (i, j)

n is total number of each item

$$E_{ij} = n \int_a^b f(x) dx = n[F(b) - F(a)] \quad (30)$$

Where $F(a) = \int_{-\infty}^a f(x) dx$

Or $E_{ij} = n[F(Y_{ij}) - F(Y_i)]$

where $F(Y_{ij}) = 1 - e^{-\left[\frac{D}{B}\right]^\alpha}$; $D > 0$ (31)

3. Analysis and result

3.1 Estimation of demand rate of multi-item

Data on monthly demand rate were obtained from Great Possibilizer Ltd. on selected items: Cowbell milk, Milo, Cerelac, SMA and Golden Morn. The demand rate of these items were modeled as appropriate probability distributions using chi-square goodness-of-fit test. Eqs (19), (22) and (28) were used to obtain the estimated average demand of items as the location parameters of normal distribution; $\hat{\mu}_M=536.92$ and $\hat{\mu}_{SMA}=10.5$ for Milo and SMA respectively, lognormal distribution; $\hat{\mu}_c = 2.1926$ and $\hat{\mu}_{GM} = 5.31$ for Cerelac and Golden Morn respectively while Weibull probability distributions was used to modeled the demand rate of cowbell milk with mean; $\hat{\mu}_{CB} = 491.65$. Easy fit software (5.6) was also used to obtain the mean of the appropriate probability distributions to validate earlier results in Equations (19), (22) and (28) as shown in Table 1.

Table 1: Summary of probability modeling, goodness-of-fit test with rank and mean estimate (average demand) for each item

Item	Probability distributions	Chi-square ranks	Average demand
Cowbell	Weibull	1	491.55
Milo	Normal	8	536.92
SMA	Normal	6	10.5
Cerelac	Lognormal	3	2.1926
Golden morn	Lognormal	5	5.3103

The calculated estimates of the parameters in eqs (19), (22) and (28) were equal or very close to the estimates obtained for same parameters with Easyfit (6.5) software. **The chi-square rank of the probability functions in column 3 of Table 1 shows the goodness-of-fit levels of the probability distribution of the demand of items. For instance, Cowbell was found to follow Weibull distribution with chi-square goodness-of-fit test of rank 1with average demand of 491.65. Milo was found to follow normal distribution with chi-square goodness-of-fit test of rank 8 with average demand of 536.92. SMA was found to follow normal distribution with chi-square goodness-of-fit test of rank 6 with average demand of 10.5, while Cerelac was found to follow lognormal distribution with chi-square goodness-of-fit test of rank 3 with average demand of 2.1926 and Golden Morn was found to follow lognormal distribution with chi-square goodness-of-fit test of rank 5 with average demand of 5.3103. It was noted that probability distribution**

with ranks 1 to 7 for Milo, ranks 1 to 5 for SMA, ranks 1 and 2 for Cerelac and ranks 1 to 4 for Golden Morn were found to be extreme distributions which do not represent demand curve of the items and were therefore discarded. Hence, the estimates obtained with the help of the software in Table 1 would be used for further analysis in this work for calculation accuracy and precision purposes.

3.2 Determination of optimal EOQ for selected items subject to the constraints

The EOQ models of section 2.1 were considered and appropriate costs and demand data obtained from the company were applied to eqs (7), (10), (12) and (14). The desired value of the nonnegative Lagrange constant (λ) was obtained for each selected item by the trial and error method in [24]. Table 2 provides the calculated EOQ subject to each constraint for $\lambda = 0$ and 1. For instance, the EOQ values for Cowbell milk when $\lambda = 0$ under warehouse, capital investment, average inventory level and number of orders constraints are the same and equal to 38. While the EOQ values for the same Cowbell milk when $\lambda=1$ under the same set of constraints: warehouse, capital investment, average inventory level and number of orders are respectively 37, 4, 38 and 39. Similarly, the EOQ values for Milo, SMA, Cerelac and Golden Morn when $\lambda = 0$ and 1 under the same set of constraints were obtained.

Table 2. EOQ values under each constraints for results for $\lambda = 0$ and 1

Item	EOQ values subject to constraint				
	λ	Warehouse	Capital Investment	Avg. inventory level	No. of Orders
Cowbell	0	38	38	38	38
	1	37	4	38	39
Milo	0	38	38	38	38
	1	38	4	38	42
SMA	0	8	8	8	8
	1	8	0.8	8	8
Cerelac	0	7	7	7	7
	1	7	0.6	6	8
Golden Morn	0	4	4	4	4

	1	4	0.4	4	10
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Table 2 shows that the calculated values of EOQ for selected items under each constraint for $\lambda=1$ is consistent for each item and is considered optimal in the sense of minimization except for number of orders constraint which requires the maximum number of orders. These values are the acceptable results and would be considered for further analysis in this work.

3.3 Optimal capacity/level of constraints for the selected items

The optimal EOQ values for individual items under each constraint in Table 2 were used to obtain the overall capacity of each constraint for the five selected items as follows:

(a) Available warehouse space constraint capacity

The warehouse space requirement is obtained from eq (8) as:

$$\sum_{i=1}^n f_i Q_i = W$$

$$\sum_{i=1}^5 f_i Q_i^* = (1.85)(37) + (0.95)(38) + (0.89)(8) + (0.17)(7) + (2.67)(4) = 124$$

The optimal storage capacity for the considered items (cowbell, milo, SMA, Cerelac and Golden Morn) is 124sq.ft out of a total space of 650 sq.ft.

(b) Investment level capacity constraint

The investment level for the selected items is obtained from eq (11) as:

$$\sum_{i=1}^n C_i Q_i = F$$

$$\sum_{i=1}^5 C_i Q_i^* = 9840(4) + 7422.41(4) + 3240.05(0.8) + 1386.37(0.6) + 10495.20(0.4) = 76,671$$

The total investment by the company for all products over the period was put at 310,000 naira. However, the calculated optimal amount of investment for the five items under consideration is obtained as 76,671 naira.

(c) Average inventory level capacity constraint

The average inventory level is obtained from eq (14) as:

$$\frac{1}{2} \sum_{i=1}^5 Q_i = M$$

$$= \frac{1}{2} \{38 + 38 + 8 + 6 + 4\} = 94 \text{ units per month}$$

Hence, the optimal stock level of the five items is 94 units per month.

(d) Numbers of orders for each item per month constraint

Recall Eq (17): $\sum_{i=1}^5 \frac{B_i}{D_i} = A$

The values of A which denotes maximum value of order for the respective items were obtained from eq (17) as follows:

$$A_{cowbel} = \frac{17835}{491.55} = 36$$

$$A_{milo} = \frac{19329}{536.92} = 36$$

$$A_{SMA} = \frac{378}{10.5} = 36$$

$$A_{cerelac} = \frac{79}{2.1926} = 36$$

$$A_{Golden\ morn} = \frac{191}{5.31} = 36$$

The results of the maximum number of order per month for each item, A=36 is a constant and when divided by the period in months (36) for the three years under consideration yield the value of 1. This implies a monthly cycle order of 1 for each item.

4. Discussion

The demand rate of the selected items follow different probability distributions. The chi-square goodness-of-fit test was used to identify the appropriate probability distributions for demand rate of the items. Both the Easyfit software (5.6) and manual computations were used to perform the goodness-of-fit test of these items as well as the parameters estimates of the distributions and results from both approaches were comparable. However, the set of results from Easyfit software was used for purposes of accuracy and precision, namely: Cowbell was found to follow Weibull distribution with chi-square goodness-of-fit test of rank 1, having shape parameter ($\alpha_{cw} =$

1.8838), scale parameter ($\beta_{cw} = 553.9$) and mean demand $E(CW) = 491.65$. Milo was found to follow normal distribution with chi-square goodness-of-fit test of rank 8, having mean (μ_M) = 536.92 and standard deviation (σ_M) = 268.28. Similarly, SMA was found to follow normal distribution with chi-square goodness-of-fit test of rank 6 having mean (μ_M) = 10.5 and standard deviation (σ_M) = 5.207, while Cerelac was found to follow lognormal distribution with chi-square goodness-of-fit test of rank 3 having mean (μ_M) = 2.1926 and standard deviation (σ_M) = 0.61301 and golden morn was also found to follow lognormal distribution with chi-square goodness-of-fit test of rank 5, having mean (μ_M) = 5.3103 and standard deviation (σ_M) = 0.66794. It was noted that probability distribution with ranks 1 to 7 for Milo, ranks 1 to 5 for SMA, ranks 1 and 2 for Cerelac and ranks 1 to 4 for Golden Morn were found to be extreme distributions that do not represent demand curve and were therefore discarded. These results are in line with [4], [15], [21] and [23] which characterized the demand rate of items and the stock out probabilities for single inventory control model with different probability distributions such as normal, uniform and Weibull.

The estimates of the location parameters of the appropriate probability distribution for the demand of each item was used as the mean demand rate to calculate the respective EOQ values. Also, the resulting EOQs were used to obtain the optimal level/capacity (for the selected items) of warehouse, investment, average inventory level and the number of orders constraints. The capacity of the warehouse space constraint for the selected items has an optimal storage capacity of 124sq.ft with a remainder of 526sq.ft for other items in the warehouse not considered in this study. More so, the investment level constraint which has the sum of 76,671 naira signifies that the optimal amount for investment of the selected five items is 76,671 naira while the remainder of 283,328.48 naira is meant for investment in other items not considered in this work. Also, the average inventory level constraint has total level (capacity) for the selected five items to be 94 units per month and the constraint of number of orders indicates that optimal number of monthly order is 1 so as to avoid shortage of items in stock.

5. Conclusion

The results obtained in this work show that the demand rate of items follow different probability distributions and the average demand rate of each item obtained as the location parameter

estimate of the respective probability distribution provides a dynamic and realistic description of the behavior and value of the demand rate over a period of time. **These estimated demand values were used to determine the dynamic optimal EOQ models for each item under consideration. It was also possible to determine the unknown specific values of the level/capacity of each constraint for the selected items as a fraction of the total capacity. This is against the use of the total capacity of each constraint for all items in stock as in [24].** Therefore, the use of probability distribution to model demand rates of items for dynamic inventory control models as against the use of simple averages in deterministic models is recommended as demand is not constant but a variable over time.

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