Original Research Article

CONSTRAINED STOCHASTIC INVENTORY CONTROL MODELS FOR MULTI-ITEM WITH VARIABLE DEMAND

Abstract

Constrained multi-item inventory system with variable demands are considered. The demand rates of five selected multi-item - Cowbell, Milo, SMA, Cerelac and Golden Morn were modeled as Weibull, Normal and Lognormal probability distributions respectively with the aid of chi-squared multinomial goodness-of-fit test. The respective probability distributions with estimated location parameters: 491.55, 536.92, 10.5, 2.1926 and 5.3103 were used as the basis of probabilistic inventory models to obtain dynamic EOQ for each item under each constraint subject to Kuhn, Karush and Tucker (KKT) conditions as against the use of simple averages in deterministic inventory models. The optimal values of these constraints: available warehouse space (124sq.ft), specified level of inventory (94 units), limited capital (76,671.52 naira) and number of orders (1/month) were obtained using the optimal EOQ values to establish optimal inventory level for each item in order to avoid shortage or excess stock.

Keywords: Constrained inventory model, multi-item, variable demand, optimal inventory, probability distribution, variable demand.

1. Introduction

Possessing a high amount of inventory for a long period of time is not usually good for a business because of inventory storage, obsolescence and spoilage costs. However, possessing too little inventory isn't good either, because the business runs the risk of losing out on potential sales and potential market share as well. The economic order quantity (EOQ) assumes that demand occurs at known constant rate and supply fulfill the replenishment order after a fixed lead time. Unfortunately, the real world is not as ideal as that. In reality, demand rate is rarely constant and hard-to-predict market is common in most practical situations. Therefore, inventory management forecasts and strategies, such as a just-in-time inventory system can be deterministic or probabilistic. According to [1], probabilistic EOQ model is an inventory model that is close to the real situation that retailers face because demand will varies from time to time. This probabilistic inventory model will incorporate the variation of the demand and uncertain lead time. Demand variation will cause a shortage especially during lead time when retailer only has a limited amount of goods to cover the demand during lead time and the goods ordered have not arrived yet. Based on that situation, there are three possibilities that can happen to the probabilistic inventory model: The first one is when demand during lead time is constant but the lead time itself varies. The second is when lead time is constant but demand during lead time varies, and the last possibility is when both lead time and demand during lead time vary.

The perspective of probabilistic continuous review inventory models with constant units of cost and lead-time demand as a random variable was presented by [2]. He gave heuristic approximate treatment for each of the backorders and the lost sales cases, while [3] studied the probabilistic single- item, single source (SISS) inventory system with zero lead-time, using classical optimization. Authors in [4] treated multi-item inventory system with budgetary constraint comparison between the Lagrange and the fixed cycle approach under the Kuhn, Karush and Tucker (KKT) conditions. These conditions were originally named after Harold W. Kuhn and Albert W. Tucker, who first published the conditions in 1951 but was later discovered that the necessary conditions for this problem had been stated by William Karush in his Master's thesis in 1939. In a later development, [5] considered both deterministic and probabilistic versions of power demand patterns with a variable rate of deterioration, while [6] considered two types of holding cost variation: (a) a nonlinear function of storage time and (b) a nonlinear function of storage level. According to [7], inventory control problems in real world usually involve multiple products which are often necessary for inventory holding thousands of items. The difficulties encountered in the practice of inventory control management were examined and it was concluded that a large gap exists between theory and practice in inventory management. In [8], a multi-item probabilistic inventory model that considered expiration factor, all unit discount policy and warehouse capacity constraints was considered. The characteristics considered in this study were probabilistic demand, perishable products, and warehouse constraints for multi-item inventory models. These conditions occur in several industries (example are companies that produce food, food sales agents, and retail goods to end customers) that consider perishable factors and warehouse constraints. The Karush-Kuhn-Tucker condition approach was used to solve the warehouse capacity problem to find the optimum point of a constrained function. The results yielded two optimal ordering times, namely ordering time-based on warehouse capacity and joint order time. A realistic and general single period for multi-item with budgetary and floor or shelf space constraints, where demand of item follow uniform probability distribution was developed by [9]. Also, [10] developed a multi-item inventory control model with instantaneous supply where demand is deterministic and follows uniform distribution for perishable items. The use of KKT conditions was employed by [11] to solve a multi-item inventory model with shortages and demand dependent on unit cost with storage space and set up cost constraints. The cost parameters were treated as fuzzy variables because of its imprecise nature while a multi-item multi-period inventory control model for known-deterministic variable subject to limited available budget was formulated by [12]. Shortages in combination with backorder, unit discount and lost sale were considered. The model was formulated into a fuzzy multi-criteria decision making (FMCDM) framework represented as a mixed integer nonlinear programming problem with the objectives to minimize both the total inventory cost and the required storage space.

Furthermore, [13] proposed a new general probabilistic multi-item, single source inventory model with varying mixture shortage cost under restrictions on backorder cost and expected varying lost sales cost. Furthermore, individual and joint replenishment policies which consisted of several products where the demands for these products followed Gamma distribution were developed by [1]. The objective was to determine the optimal ordering quantity that minimizes the total cost for each product, and [14] examined the probability distributions of variable demand rate of multi-item inventory problem. The result showed that demands of selected products follow certain probability distributions namely; normal, uniform and Weibull distributions. Optimal order quantities and the probabilities of shortage and no shortage were also obtained for the selected products. The multi objective particle swarm optimization (MOPSO) was utilized by [15] to solve a multi item inventory control model which was developed to optimize the total inventory cost and inventory layout management using a metaheuristic algorithm named multi-objective particle swarm optimization (MOPSO) algorithm.

On this perspective, we sought to obtain a probabilistic multi-item inventory model by modeling the variable demand rate using appropriate probability distribution functions. The respective probability distribution functions shall be used as the basis to obtain dynamic EOQ values for items subject to the constraints under the Kuhn-Tucker conditions.

1.2 Problem definition and Assumptions

A distributor company of multi-item gets its supply from different manufacturing companies at different times by placing an order to the manufacturing companies. The company is faced with several restrictions that limits her capacity to make orders at will and convenience. These restrictions (constraints) include total available warehouse space of 650sq.ft and total limited capital of 310,000 naira among others. However, only five items were selected out of various multi-items in the warehouse for study with warehouse space, required capital, level of inventory and number of orders constraints as unknown. It therefore requires a robust and dynamic approach of estimating the demand of each of the selected item to enable her determine the ordering frequency and the best quantity to be ordered for each product at appropriate time to avoid shortage or excess stock within the sphere of the stated conditions (constraints). The capacities/levels of the unknown constraints for the selected items need be obtain to determine what quantities are left for the other items not considered in the study. Under this consideration, demand rates of an item is assumed to follow a probability distribution. The probability of lead time is between zero and one; Pr[$0 \leq LT \leq 1$], shortages are not allowed, purchase cost and reordering costs do vary with time for a specified period.

2. Methodology

2.1 Formulation of EOQ Model with constraints

Notations:

TVC = Total variable cost

n = total number of items being controlled simultaneously

f_i = floor area (storage space) required per unit of item i (i = 1,2, …, n)

W = warehouse space limit to store all items in the inventory.

λ = non-negative lagrange multiplier

D_i = annual demand for ith item

$\overline{D_i}$ = average demand for each item (i = 1,2,3, …, n)

Q_i = order quantity for items i in inventory (i = 1, 2, ..., n)

M = upper limit of average number of units for all items in the stock

C_i = price per unit of items in the stock

F = investment limit for all items in the inventory

C_{oi} = order cost per order (i = 1,2,3, …, n)

C_{hi} = cost of carrying one unit of an item in the inventory for a given length of time

A = Maximum value of order

B_i = Total quantity ordered for each item (i = 1,2,3, …, n)

S = Maximum shortage quantity

Let the decision variables be D_i, Q_i, C_{oi} *and* C_{hi}. The objective function as adapted from [16] is:

$$\text{Min TVC} = \sum_{i=0}^{n}\left[\frac{D_i}{Q_i}C_{oi} + \frac{Q_i}{2}C_{hi}\right] \tag{1}$$

The constraints:

1. Warehouse space availability:

$$\sum_{i=0}^{n} f_i\, Q_i \le \text{W} \tag{2}$$

2. Capital limited:

$$\sum_{i=0}^{n} C_i\, Q_i \le \text{F} \tag{3}$$

3. Inventory level specification:

$$\frac{1}{2}\sum_{i=0}^{n} Q_i \le \text{M} \tag{4}$$

4. Order quantities:

$$\sum_{i=0}^{n}\frac{B_i}{Q_i} \le A \tag{5}$$

5. Non negativity:

$$D_i, Q_i, C_{oi} \text{ and } C_{hi} \ge 0$$

The models are:

1. Min TVC = $\sum_{i=0}^{n}\left[\frac{D_i}{Q_i}C_{oi} + \frac{Q_i}{2}C_{hi}\right]$

   Subject to: $\sum_{i=0}^{n} f_i\, Q_i \le \text{W}$

2. Min TVC = $\sum_{i=0}^{n}\left[\frac{D_i}{Q_i}C_{oi} + \frac{Q_i}{2}C_{hi}\right]$

   Subject to: $\sum_{i=0}^{n} C_i \; Q_i \; \leq$ F

3. Min TVC = $\sum_{i=0}^{n}\left[\frac{D_i}{Q_i}C_{oi} + \frac{Q_i}{2}C_{hi}\right]$

   Subject to: $\frac{1}{2}\sum_{i=0}^{n} Q_i \quad \leq$ M

4. Min TVC = $\sum_{i=0}^{n}\left[\frac{D_i}{Q_i}C_{oi} + \frac{Q_i}{2}C_{hi}\right]$

   Subject to: $\sum_{i=0}^{n}\frac{B_i}{Q_i} \quad \leq A$

   D_i, Q_i, C_{oi} *and* $C_{hi} \geq 0$

By applying the KKT necessary and sufficient condition for optimal value of TVC, we obtain the following formulations with respect to each constraint:

1. Warehouse space available:

$$\sum_{i=0}^{n} f_i \; Q_i \;\; \leq \text{W}$$

If the warehouse space required for each unit of item, i is f_i (i=1,2, … ,n), then the total storage area (or volume) required by all n inventory items must be less than or equal to the total available space storage area (or volume) of the warehouse. This constraint indicates that even if all items reach their maximum inventory levels simultaneously, the warehouse space should be sufficient to store the inventory of these items with an assumption that all the five items are received together. Thus, the problem is to minimize the total variable inventory cost for each item under warehouse constraint.

$$L(Q_i, \lambda) = \text{TVC} + \lambda\left[\sum_{i=1}^{n} f_i Q_i - W\right] \tag{6}$$

$$= \sum_{i=0}^{n}\left[\frac{D_i}{Q_i}C_{oi} + \frac{Q_i}{2}C_{hi}\right] + \lambda\left[\sum_{i=1}^{n} f_i Q_i - W\right]$$

The necessary condition for L to be minimum is:

$$\frac{\partial L}{\partial Q_i} = \frac{\partial}{\partial Q_i}\left[\sum_{i=0}^{n}\left[\frac{D_i}{Q_i}C_{oi} + \frac{Q_i}{2}C_{hi}\right] + \lambda\left[\sum_{i=1}^{n} f_i Q_i - W\right]\right] = 0$$

$$= -\frac{2D_iC_{oi}}{Q_i^2} + \frac{C_{hi}}{2} + \lambda f_i = 0$$

$$\frac{-2D_iC_{oi} + Q_i^2\; C_{hi} + 2\lambda f_i Q_i^2}{2Q^2{}_i} = 0$$

$$-2D_iC_{oi} = -Q_i^2\left[C_{hi} + 2\lambda f_i\right]$$

$$Q_i^2 = \frac{2D_iC_{oi}}{C_{hi} + 2\,\lambda f_i}$$

$$Q_i^* = \sqrt{\frac{2D_iC_{oi}}{C_{hi}+2\lambda f_i}}\,; \quad i = 1,2,\dots,n \tag{7}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} f_i\, Q_i - W = 0$$

$$\sum_{i=1}^{n} f_i\, Q_i = \mathrm{W}; \quad \text{since } \lambda \geq 0 \tag{8}$$

2. Limited capital:

$$\sum_{i=0}^{n} C_i\, Q_i \leq \mathrm{F}$$

Since investment on inventory is substantial for many organizations, decision makers must put a restriction on the amount of inventory to be carried. The inventory control policy is accordingly adjusted to achieve the objective of keeping total investment required within limit. Hence, the problem is to minimize the total variable inventory cost for each item under the investment constraint, along with the assumption that both demand and lead time are constant and known.

$$L(Q_i, \lambda) = \mathrm{TVC} + \lambda\Big[\sum_{i=1}^{n} C_iQ_i - F\Big] \tag{9}$$

$$= \sum_{i=0}^{n}\left[\frac{D_i}{Q_i}C_{oi} + \frac{Q_i}{2}C_{hi}\right] + \lambda\Big[\sum_{i=1}^{n} C_iQ_i - F\Big]$$

$$\frac{\partial L}{\partial Q_i} = \frac{\partial}{\partial Q_i}\left[\sum_{i=0}^{n}\left[\frac{D_i}{Q_i}C_{oi} + \frac{Q_i}{2}C_{hi}\right] + \lambda\Big[\sum_{i=1}^{n} C_iQ_i - F\Big]\right] = 0$$

$$= -\frac{D_iC_{oi}}{Q_i^2} + \frac{C_{hi}}{2} + \lambda C_{oi} = 0$$

$$-2D_iC_{oi} + Q_i^2C_{hi} + 2\lambda Q_i^2C_{oi} = 0$$

$$Q_i^2 = \frac{2D_iC_{oi}}{C_{hi} + 2\lambda C_{oi}}$$

$$Q_i^* = \sqrt{\frac{2D_iC_{oi}}{C_{hi}+2\lambda C_i}}\,; \quad i = 1,2,\dots,n \tag{10}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=1}^{n} C_i\, Q_i - F = 0$$

$$\sum_{i=1}^{n} C_i\, Q_i = F; \quad \text{since } \lambda \geq 0 \tag{11}$$

3. Inventory level specification:

$$\frac{1}{2}\sum_{i=0}^{n} Q_i \leq M$$

Since the average number of units in the inventory of an item, i is $Q_i/2$, and it is required that the average number of units of individual items held together in the inventory should not exceed the prespecified number, M. The problem is to minimize the total variable inventory cost subject to the limitation of total average inventory level of items.

$$L(Q_i, \lambda) = \text{TVC} + \lambda \left[\frac{1}{2}\sum_{i=1}^{n} Q_i - M\right] \quad (12)$$

$$= \sum_{i=0}^{n} \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi}\right] + \lambda \left[\frac{1}{2}\sum_{i=1}^{n} Q_i - M\right]$$

$$\frac{\partial L}{\partial Q_i} = \frac{\partial}{\partial Q_i} \left[\sum_{i=0}^{n} \left[\frac{D_i}{Q_i} C_{oi} + \frac{Q_i}{2} C_{hi}\right] + \lambda \left[\frac{1}{2}\sum_{i=0}^{n} Q_i - M\right]\right] = 0$$

$$= -\frac{D_i C_{oi}}{Q_i^2} + \frac{C_{hi}}{2} + \frac{\lambda}{2} = 0$$

$$-2D_i C_{oi} + Q_i^2 [C_{hi} + \lambda] = 0$$

$$Q_i^2 = \frac{2D_i C_o}{C_{hi} + \lambda}$$

$$Q_i^* = \sqrt{\frac{2D_i C_{oi}}{C_{hi} + \lambda}}; \quad i = 1, 2, \dots, n \quad (13)$$

$$\frac{\partial L}{\partial \lambda} = \frac{1}{2}\sum_{i=1}^{n} Q_i - M = 0$$

$$\frac{1}{2}\sum_{i=1}^{n} Q_i = M; \quad \text{since } \lambda \geq 0 \quad (14)$$

4. Orders quantity:

$$\sum_{i=0}^{n} \frac{B_i}{Q_i} \leq A$$

Number of orders is very important in inventory as it eliminates the existence of shortage and excess. This additional constraint helps us to determine the minimum number of monthly orders where shortages is zero. A TVC was formulated by [17] by incorporating purchase with shortage and production with shortage. Our aim is to obtain the number of inventory orders that should be considered. This number of order constraint will help us determine how many times or batches should items be ordered so as to minimize cost. The required EOQ model when the constant number of orders is active is obtained as follows:

$$L(Q_i, \lambda) = \text{TVC} + \lambda \left[\sum_{i=0}^{n} \frac{B_i}{Q_i} - A\right] \quad (15)$$

$$L(Q_i, \lambda) = \frac{C_{oi} D_i}{Q_i} + \frac{C_{hi}(Q_i - S)^2}{2Q_i} + P_i \frac{S^2}{2Q_i} + \lambda \left[\sum_{i=0}^{n} \frac{B_i}{Q_i} - A\right]$$

$$\frac{\partial L}{\partial Q_i} = \frac{\partial}{\partial Q_i} \left[-\frac{C_{oi} D_i}{{Q_i}^2} + \frac{C_{hi}}{2} - \lambda \frac{B_i}{{Q_i}^2}\right] = 0, \; s = 0$$

$$-\,C_{oi}D_i + \frac{{Q_i}^2 C_{hi}}{2} - \lambda B_i \;= 0$$

$$-\,2C_{oi}D_i + \;{Q_i}^2 C_{hi} - 2\lambda B_i = 0$$

$${Q_i}^2 C_{hi} = 2C_{oi}D_i + 2\lambda B_i$$

$${Q_i}^2 = \;\frac{2C_{oi}D_i + 2\lambda B_i}{C_{hi}}$$

$${Q_i}^* = \sqrt{\frac{2C_{oi}D_i + 2\lambda B_i}{C_{hi}}}\;; \qquad i \;=\; 1,2, \ldots, n \tag{16}$$

$$\frac{\partial L}{\partial \lambda} = \sum_{i=0}^{n} \frac{B_i}{Q_i} \;\; - A \;= 0$$

$$\sum_{i=0}^{n} \frac{B_i}{\overline{D}_i} \;\;= A \qquad \text{(where } Q_i \;=\; \overline{D}_i\text{);} \;\;\; \text{since } \lambda \geq 0 \tag{17}$$

2.2 Probability distribution of demand rate of items

In reality, demand will vary from time to time. Thus, probabilistic inventory model will incorporate the variation of demand and uncertain lead time. In this case, demand of an item is not known to be deterministic but are variables. Hence, the need to obtain the probability distributions for the demand of the multi-item. We shall consider the following cases after necessary preliminary analysis:

1. When Demand of item is assumed to follow normal distribution.

2. When demand of item is assumed to follow Weibull distribution.

3. When demand of item is assumed to follow lognormal distribution.

2.2.1 Probability inventory model when demand follows a normal distribution

The probability distribution of the demand of the products were not known and were assumed to be normally distributed with the density function:

$$f_D(\mathrm{D}, \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}} e^{\frac{(D-\mu)^2}{2\sigma^2}} \;;\; \mathrm{D} \geq 0, \sigma\, > 0 \tag{18}$$

Where D is the demand (random variable), μ and σ^2 are mean and variance of the normal distribution.

Estimation of parameters of normal distribution for quantity demand using Maximum Likelihood method

Let $L(\mu, \sigma^2;\ D_1, D_2, \ldots\ D_n)$ and $\ln(\ l\,f(\mu, \sigma^2;\ D_1, D_2, \ldots\ D_n)$ be the likelihood and loglikelihood functions.

By substituting (18), we have:

$$\ln \prod_{i=1}^{n}\left[(2\pi\sigma^2)^{-\frac{1}{2}} \exp\left(-\frac{1}{2\sigma^2}(D_i - \mu)^2\right)\right] = \ln\left[(2\pi\sigma^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(D_i - \mu)^2\right)\right]$$

$$\ln(2\pi\sigma^2)^{-\frac{n}{2}} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(D_i - \mu)^2 = -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(D_i - \mu)^2$$

$$\frac{\partial \boldsymbol{L}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2;\ \boldsymbol{D}_1, \boldsymbol{D}_2, \ldots\ \boldsymbol{D}_n)}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(D_i - \mu)^2 = 0$$

$$\sum_{i=1}^{n} D_i - n\mu = 0$$

$$\therefore \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n} D_i \qquad (19)$$

$$\frac{\partial \boldsymbol{L}(\boldsymbol{\mu}, \boldsymbol{\sigma}^2;\ \boldsymbol{D}_1, \boldsymbol{D}_2, \ldots\ \boldsymbol{D}_n)}{\partial \boldsymbol{\sigma}^2} = -\frac{n}{2}\left(\frac{2\pi}{2\pi\boldsymbol{\sigma}^2}\right) + \frac{1}{2(\boldsymbol{\sigma}^2)^2}\sum_{i=1}^{n}(D_i - \mu)^2 = 0$$

$$-\frac{2\pi n}{4\pi\boldsymbol{\sigma}^2} + \frac{1}{2(\boldsymbol{\sigma}^2)^2}\sum_{i=1}^{n}(D_i - \mu)^2 = -\frac{n}{2\boldsymbol{\sigma}^2} + \frac{1}{2(\boldsymbol{\sigma}^2)^2}\sum_{i=1}^{n}(D_i - \mu)^2 = 0$$

$$-n + \frac{1}{\boldsymbol{\sigma}^2}\sum_{i=1}^{n}(D_i - \mu)^2 = -\boldsymbol{\sigma}^2 + \frac{1}{\boldsymbol{n}}\sum_{i=1}^{n}(D_i - \mu)^2 = 0$$

$$\therefore \widehat{\boldsymbol{\sigma}^2} = \frac{1}{\boldsymbol{n}}\sum_{i=1}^{n}(D_i - \mu)^2 \qquad (20)$$

2.2.2 Probability inventory model for demand with lognormal distribution

A continuous random variable D is said to have a lognormal distribution with mean μ and variance σ^2 if the density function is given by:

$$f_D(\ln(D), \mu, \sigma^2) = \frac{1}{\sigma\sqrt{2\pi}}\, e^{\frac{-(\ln(D)-\mu)^2}{2\sigma^2}} \ ;\ D_i \geq 0, \sigma > 0 \qquad (21)$$

Where D is the quantity demand (random variable), μ and σ^2 are the mean and variance of the quantity demand**.**

Parameters Estimation

We use the maximum likelihood method of parameter estimation as follows:

Let $l(\mu,\ \sigma^2;\ \ln(D_1), \ln(D_2), \ldots\ \ln(D_n))$ and $\ln[\mu,\ \sigma^2;\ \ln(D_1), \ln(D_2), \ldots\ \ln(D_n)]$ be the likelihood and log likelihood functions.

By substituting (21):

$$\ln\left[(2\pi\sigma^2)^{-\frac{n}{2}}\exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln D_i - \mu)^2\right)\right] = \ln(2\pi\sigma^2)^{-\frac{n}{2}} - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln D_i - \mu)^2$$

$$= -\frac{n}{2}\ln(2\pi\sigma^2) - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(\ln D_i - \mu)^2$$

$$\frac{\partial L(\boldsymbol{\mu},\boldsymbol{\sigma}^2;\boldsymbol{D}_1,\boldsymbol{D}_2,\dots\boldsymbol{D}_n)}{\partial\mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(\ln D_i - \mu)^2 = 0$$

$$\sum_{i=1}^{n}\ln D_i - n\mu = 0$$

$$\therefore \hat{\mu} = \frac{1}{n}\sum_{i=1}^{n}\ln D_i \quad (22)$$

$$\frac{\partial L(\boldsymbol{\mu},\boldsymbol{\sigma}^2;\ln(\boldsymbol{D}_1),\ln(\boldsymbol{D}_2),\dots\ln(\boldsymbol{D}_n))}{\partial\boldsymbol{\sigma}^2} = -\frac{n}{2}\left(\frac{2\pi}{2\pi\sigma^2}\right) + \frac{1}{2(\sigma^2)^2}\sum_{i=1}^{n} ln(D_i - \mu)^2 = 0$$

$$-\frac{2\pi n}{4\pi\boldsymbol{\sigma}^2} + \frac{1}{2(\boldsymbol{\sigma}^2)^2}\sum_{i=1}^{n}[\ln(D_i) - \mu]^2 = -\frac{n}{2\boldsymbol{\sigma}^2} + \frac{1}{2(\boldsymbol{\sigma}^2)^2}\sum_{i=1}^{n}[\ln(D_i) - \mu]^2 = 0$$

$$-n + \frac{1}{\boldsymbol{\sigma}^2}\sum_{i=1}^{n}[\ln(D_i) - \mu]^2 = -\boldsymbol{\sigma}^2 + \frac{1}{\boldsymbol{n}}\sum_{i=1}^{n}[\ln(D_i) - \mu]^2 = 0$$

$$\therefore \hat{\boldsymbol{\sigma}}^2 = \frac{1}{\boldsymbol{n}}\sum_{i=1}^{n}[\ln(D_i) - \mu]^2 \quad (23)$$

2.2.3 Probability inventory model for demand with Weibull distribution

The probability density function of a two parameter Weibull random variable is;

$$f(D,\alpha,\beta) = \begin{cases} \left(\frac{\alpha}{\beta}\right)(D)^{\alpha-1}\, e^{-\left[\frac{\bar{D}}{\beta}\right]^{\alpha}}; & D>0 \\ 0, & if\ \ D \le 0 \end{cases} \quad (24)$$

Estimation of parameters

We use the least squares method of estimation as follows:

$$f(D,\alpha,\beta) = \left(\frac{\alpha}{\beta}\right)D^{\alpha-1}\ \prod_{i=1}^{n}\left(\frac{D}{\beta}\right)^{\alpha-1}\ e^{\sum_{i=1}^{n}\left[\frac{\bar{D}}{\beta}\right]^{\alpha}}$$

$$F(D,\alpha,\beta) = 1 - e^{\left[\frac{\bar{D}}{\beta}\right]^{\alpha}} \quad (25)$$

$$\ln\left[\frac{1}{1-F(D)}\right] = \alpha\ln D - \alpha ln\beta \quad (26)$$

Let D_i be a random sample of the demand and $F(D)$ is estimated and replaced by the median rank method as follows:

$$F(D) = \frac{i-0.3}{n+0.4}, (D_i,\ i = 1,2,\dots,n) and\ (D_1 < D_2 < \dots < D_n)$$

Eq (26) becomes;

$Y = \alpha X + \lambda$

Where $Y = \ln[-ln(1 - F(D))]$ (27)

$X_i = lnD \; and \; \lambda = -\alpha ln\beta$

Estimates of $\alpha \; and \; \beta$ can be obtained in (27) as:

$$\hat{\alpha} = \frac{\left[\sum_{i=1}^{n} XY - \frac{1}{n}\sum_{i=1}^{n} X \;\; \sum_{i=1}^{n} Y\right]}{\sum_{i=1}^{n} X^2 - \frac{1}{n}(\sum_{i=1}^{n} X)^2}$$

$$\hat{\lambda} = \frac{1}{n}\sum_{i=1}^{n} Y - \frac{\alpha}{n}\sum_{i=1}^{n} X$$

$$\hat{\lambda} = \bar{Y} - \hat{\alpha}\bar{X}$$

$$\hat{\beta} = e^{-\frac{\lambda}{\alpha}}$$

The mean of Weibull distribution is given by;

$$E[D] = \beta \, \Gamma\left(\alpha + \frac{1}{\alpha}\right) \quad (28)$$

2.3 Chi-square goodness-of-fit test

The chi-square goodness-of-fit test would be used to determine how well the sample data fits a distribution from a population. It establishes the discrepancy between the observed values and expected values.

The test statistic is given by

$$\chi^2 = \sum_{i=1}^{n} \frac{(O_{ij} - E_{ij})^2}{E_{ij}} \sim \chi^2_{\alpha;n-1} \quad (29)$$

Where O_{ij} are the observed value in cell (i, j)

E_{ij} are the expected value in cell (i, j)

n is total number of each item

$$E_{ij} = n\int_a^b f(x)dx = n[F(b) - F(a)] \quad (30)$$

Where $F(a) = \int_{-\infty}^{a} f(x)dx$

or$E_{ij} = n\left[F\left(Y_{ij}\right) - F(Y_i)\right]$

where $F(Y_{ij})$ = 1 - $e^{-\left[\frac{D}{B}\right]^{\alpha}}$; $D > 0$ (31)

3. Analysis and result

3.1 **Estimation of demand rate of multi-item**

Data on monthly demand rate were obtained from Great Possibilizer Ltd. on Cowbell milk, Milo, Cerelac, SMA and Golden Morn. The demand rate of these items were modeled as appropriate probability distributions using chi-squared goodness-of-fit test. Eqns (19), (22) and (28) were used to obtain the estimated average demand of items as the location parameters of normal distribution; $\hat{\mu}_M$=536.92 and $\hat{\mu}_{SMA}$=10.5 for Milo and SMA respectively, lognormal distribution; $\hat{\mu}_c$ = 2.1926 and $\hat{\mu}_{GM}$ = 5.31for Cerelac and Golden Morn respectively while Weibull probability distributions has mean; $\hat{\mu}_{CB}$ = 491.65. Easy fit software (5.6) was also used to obtain the mean of the appropriate probability distributions to validate earlier results in Equations (19), (22) and (28) as shown in Table 1.

The calculated estimates of the parameters in eqns (19), (22) and (28) were equal or very close to the estimates obtained for same parameters with Easyfit (6.5) software. Therefore, the estimates obtained with the help of the software in Table 1 would be used for further analysis in this work for calculation accuracy and precision.

Table 1: Summary of probability modeling, goodness-of-fit test with rank and mean estimate (average demand) for each item

Item	Probability distributions	Chi-square ranks	Average demand
Cowbell	Weibull	1	491.55
Milo	Normal	8	536.92
SMA	Normal	6	10.5
Cerelac	Lognormal	3	2.1926
Golden morn	Lognormal	5	5.3103

3.2 **Determination of optimal EOQ for selected items subject to the constraints**

The EOQ models of section 2.1 were considered and appropriate costs and demand data obtained from the company were applied to eqns (7), (10), (12) and (14). The desired value of the nonnegative lagrange constant (λ) was obtained for each selected item by the trial and error method (Sharma, 2013). Table 2 provides the calculated EOQ subject to each constraint for $\lambda = 0$ and 1.

Table 2. EOQ values under each constraints for results for λ = 0 and 1

	EOQ values subject to constraint				
Item	λ	Warehouse	Capital Investment	Avg. inventory level	No. of Orders
Cowbell	0	38	38	38	38
	1	37	4	38	39
Milo	0	38	38	38	38
	1	38	4	38	42
SMA	0	8	8	8	8
	1	8	0.8	8	8
Cerelac	0	7	7	7	7
	1	7	0.6	6	8
Golden Morn	0	4	4	4	4
	1	4	0.4	4	10

Table 2 shows that the calculated values of EOQ for selected items under each constraint for λ= 1 is consistent for each item and is considered optimal in the sense of minimization except for number of orders constraint. These values are therefore chosen for further analysis in this work.

3.3 Optimal capacity/level of constraints for the selected items

The optimal EOQ values for individual items under each constraint in Table 2 was used to obtain the overall capacity of each constraint for the five selected items as follows:

(a) Available warehouse space constraint capacity

The warehouse space requirement is obtained from eqn (2):

$\sum_{i=1}^{n} f_i \, Q_i \leq W$ and $\sum_{i=1}^{n} f_i \, Q_i \; = \;$ W

$$\sum_{i=1}^{5} f_i \; Q_i^* = (1.85)(37) + (0.95)(38) + \; (0.89)(8) + (0.17)(7) + (2.67)(4) = 124$$

The optimal storage capacity for the considered items (cowbell, milo, SMA, Cerelac and Golden Morn is 124sq.ft out of a total space of 650 sq.ft.

(b) Investment level capacity constraint

The investment level for the selected items is obtained eqn (3):

$\sum_{i=1}^{n} C_i \, Q_i \leq F$ and $\sum_{i=1}^{n} C_i \, Q_i = F$

$$\sum_{i=1}^{5} C_i \; Q_i^* = 9840(4) + \; 7422.41(4) + 3240.05(0.8) + \; 1386.37(0.6) + 10495.20(0.4)$$

$$= 76{,}671$$

The total investment by the company for all products over the period was put at 310,000 naira. However, the calculated optimal amount of investment for the five items under consideration is obtained as 76,671 naira.

(c) Average inventory level capacity constraint

The average inventory level is obtained from eqn (4):

$\frac{1}{2}\sum_{i=1}^{5} Q_i \; \leq M$ and $\frac{1}{2}\sum_{i=1}^{5} Q_i \; = M$

$= \frac{1}{2}\{38 + 38 + 8 + 6 + 4\} = 94 \; units\; per\; month$

Hence, the optimal stock level of the five items is 94 units per month.

(d) Numbers of orders for each item per month constraint

Recall Eqn (5): $\sum_{i=1}^{5} \frac{B_i}{D_i} \leq A$ and $\sum_{i=1}^{5} \frac{B_i}{D_i} = A$

The values of A for the respective items were obtained as follows:

$$A_{cowbel} = \frac{17835}{491.55} = 36$$

A_{milo} = $\frac{19329}{536.92}$ = 36

A_{SMA} = $\frac{378}{10.5}$ = 36

$A_{cerelac}$ = $\frac{79}{2.1926}$ = 36

$A_{Golden\ morn}$ = $\frac{191}{5.31}$ = 36

The results, A=36 is a constant and when divided by the period in months (36) for the three years under consideration yield the value of 1. This implies a monthly cycle order of 1 for each item.

4. Discussion

The demand rate of the selected items followed different probability distributions. The chi-square goodness of fit test was used to identify the appropriate probability distributions for demand rate of the items. Cowbell was found to follow Weibull distribution with chi-square goodness of fit test of rank 1, having shape parameter (α_{cw} = 1.8838), scale parameter (β_{cw} = 553.9) and mean E(CW) = 491.65. Milo was found to follow normal distribution with chi-square goodness of fit test of rank 8, having mean (μ_M) = 536.92 and standard deviation (σ_M) = 268.28. SMA was found to follow normal distribution with chi-square goodness of fit test of rank 6 having mean (μ_M) = 10.5 and standard deviation (σ_M) = 5.2071 while Cerelac was found to follow lognormal distribution with chi-square goodness of fit test of rank 3 having mean (μ_M) = 2.1926 and standard deviation (σ_M) = 0.61301 and golden morn was found to followed lognormal distribution with chi-square goodness of fit test of rank 5, having a mean(μ_M) = 5.3103 and standard deviation(σ_M) = 0.66794. The Easyfit software (5.6) was used to confirm the goodness of fit test of these items as well as the parameters estimates of the distributions. It was noted that probability distribution with ranks 1 to 7 for Milo, rank 1 to 5 for SMA, rank 1 and 2 for Cerelac and rank 1 to 4 for Golden Morn were found to be extreme distributions that do not represent demand curve and were therefore discarded. These results are in line with [9], [14] and [18] who showed that the demand rate of multi-item for single inventory control model followed different probability distributions such as normal, uniform and Weibull distribution.

The estimates of the location parameters of the appropriate probability distribution for the demand of each item was used as the mean demand rate to calculate the respective EOQ values. Also, the resulting EOQs were used to obtain the optimal level/capacity (for the selected items) of warehouse constraint, investment constraint, average inventory level constraint and the number of orders constraint. The capacity of the warehouse space constraint for the selected items has an optimal storage capacity of 124sq.ft with a remainder of 526sq.ft for other items in the warehouse not considered in this study. The investment level constraint which has the sum of 76,671 naira signifies that the optimal amount for investment of the selected five items is 76,671 naira while the remainder of N283,328.48 is meant for investment in other items not considered in this work. The average inventory level constraint has total level (capacity) for the selected five items to be 94 units per month. The constraint of number of orders indicates that optimal number of monthly order is 1 so as to avoid shortage or excess of items in stock.

5. Conclusion

The results obtained in this work show that the demand of items follow different probability distributions and the average demand rate of each item obtained as the location parameter estimate of the respective probability distribution provides a dynamic and realistic description of the behavior and value of the demand rate over a period of time for the determination of its dynamic optimal EOQ. Therefore, the use of probability distribution to model demand rates of items for dynamic inventory control models as against the use of simple averages in deterministic models is recommended as demand is not constant but a variable over time.

References

[1] Lesmono D, Limansyah T. A multi-item probabilistic inventory model. Asian Mathematical Conference, IOP Conference Series: Journal of Physics: Conf. Series 893; 2017.

[2] Hadley G, Whitin TM. Analysis of Inventory Systems. Englewood Cliffs, NJ: Prentice Hall Inc; 1963.

[3] Fabrycky WJ, Banks J. Procurement and Inventory Systems: Theory and Analysis. New York: Reinhold Publishing Corporation, 1967.

[4] Rosenblatt BS. Modern Business: A Systems Approach 2nd Ed. Boston: Houghton Mifflin Co; 1977.

[5] Datta TK, Pal AK. Order level inventory system with power demand pattern for items with variable rate of deterioration. Indian Journal of Pure and Applied Mathematics, 1988;19:1043–1053
[6] Goh M. Some results for inventory models having inventory level dependent demand rate. International Journal of Production Economics. 1992; 27(1):155–160.

[7] Lenard JD, Roy B. Multi-item inventory control: A multi-criteria view. European Journal of Operation Research. 1995; 87: 685-692.

[8] Abou-El-Ata. Probabilistic multi-item inventory model with varying order cost under two restrictions: A geometric programming approach. International Journal of Production Economics. 2003; 83:223-231.

[9] Beb-Daya M, Darvish MA, Rahim A. Two-stage imperfect production system with inspection errors. International journal of Operations and Quantitative Management. 2005; 9(2):117–131.

[10] Purohit GN, Rathore H. Multi-Item Inventory Control Model with Space Constraints for Perishable Items. The IUP Journal of Operations Management. 2012; 11(1): 35-41.

[11] Vasanthi P, Seshaiah CV. Multi Item Inventory Model with Shortages under Limited Storage Space and Set up Cost Constraints via Karush Kuhn Tucker Conditions Approach. Applied Mathematical Sciences. 2013; 7(102): 5085-5094.

[12] Mousavi SM, Niaki ST, Bahreininejad A, Musa SN. Multi-Item Multi-periodic Inventory Control Problem with Variable Demand and Discounts: A Particle Swarm Optimization Algorithm. Scientific world Journal. 2014; 2014: 1-16.

[13] Fergany HA. Probabilistic multi-item inventory model with varying mixture shortage cost under restrictions. Springerplus. 2016; 5(1): 1351.

[14] Udoh NS. Probabilistic inventory control models with variable demand over a planning horizon for consumable products. World Journal of Applied Science and Technology. 2018; 10(1B): 37-243.

[15] Sarwar F, Rahman M, Ahmed M. A Multi-Item Inventory Control Model using Multi Objective Particle Swarm Optimization (MOPSO). Proceedings of the International Conference on Industrial Engineering and Operations Management. Bangkok, Thailand. (2019).

[16] Sharma JK. Operations Research: Theory and Applications. India: Macmillan Publishers Ltd. 2013.

[17] Gafour KM, Rashid R. Development of multi-item EOQ when the constraints of investment and storage capacity are active. *AIP Conference Proceedings*, 1691. Htpp://dx.doi.org/10.1063/1.4937024. (2016).

[18] Ould-Louly MA. A dynamic single-stage multi-item inventory control model. Large Scale Systems: Theory and Applications. 2001; 34(8):73-78.